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# Lagrangian Equation of Motion

## Purpose:

- To extend the *Energy* approach in deriving equations of motion (i.e. *Lagrange's Method*) for Mechanical Systems.

## Topics:

- Generalized Coordinates
- Lagrangian Equation of Motion for Independent Set of Generalized Coordinates
- Lagrangian Equation of Motion for Dependent Set of Generalized Coordinates
- Hamiltonian Principle



## Generalized Coordinates

**Definition:** A “*Generalized Coordinate System*” is any set of variables  $\{q^m\}$  which defines the position of a mechanical system. Generalized coordinates are usually chosen as the best description of the mechanical configuration (i.e. Polar, Cylindrical, Spherical, etc.).

**Definition:** “*Degrees-of-Freedom*” is the minimum number of generalized coordinates which adequately define the position of the mechanical system.



**Example:** {  
A free **Particle** in space has: 3-DOF  
A free **Rigid Body** in space has: 6-DOF  
A **Structure** has: 0-DOF

An analyst for convenience may choose to use more generalized coordinates than the number of degrees-of-freedom of the system. In this case, the set of generalized coordinates is not independent. Therefore:

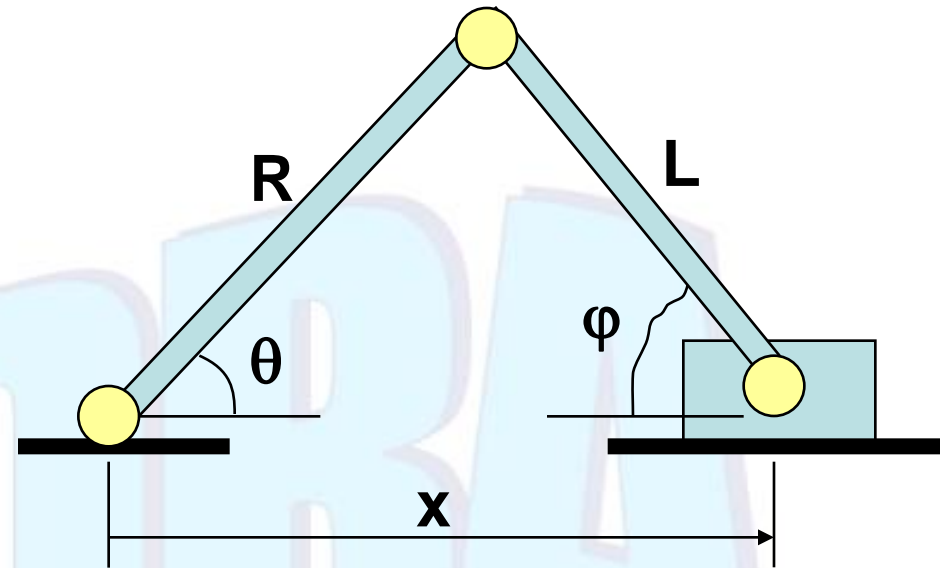
**An Independent Coordinate Set:** exists when the number of generalized coordinates correspond to the degree-of-freedom, and

**A Dependent Coordinate Set (Constraints):** exists when we have a number of functional relations among the generalized coordinates.



**Example:** Consider the one degree-of-freedom system shown.

$\{\theta\}$  : Independent Set  
 $\{\theta, \varphi, x\}$ : Dependent Set



Constraints are:

$$\begin{cases} g_1 = R \sin \theta - L \sin \varphi = 0 \\ g_2 = R \cos \theta + L \cos \varphi - X = 0 \end{cases}$$

$g_1$  and  $g_2$  functions of  $\{q^m\}$



**Definition:**  $\underline{\dot{q}}$  or  $\{\dot{q}^m\}$  is defined as the **Generalized Velocity**.

**Definition:**  $\delta q$  or  $\{\delta q^m\}$  is defined as the **Generalized Virtual Displacement**.

**Definition:** The **Generalized Momentum**  $\{P_m\}$  is the

Set  $\left\{ \frac{\partial T}{\partial \dot{q}^m} \right\}$ .

$$\{P_m\} = \left\{ \frac{\partial T}{\partial \dot{q}^m} \right\} \quad (11.1)$$

**Note that:** In the Newtonian Mechanics, the momentum of  $\beta$ th particle is:

$$P_i^\beta = \frac{\partial T}{\partial \dot{x}_i^\beta} \quad (11.2)$$



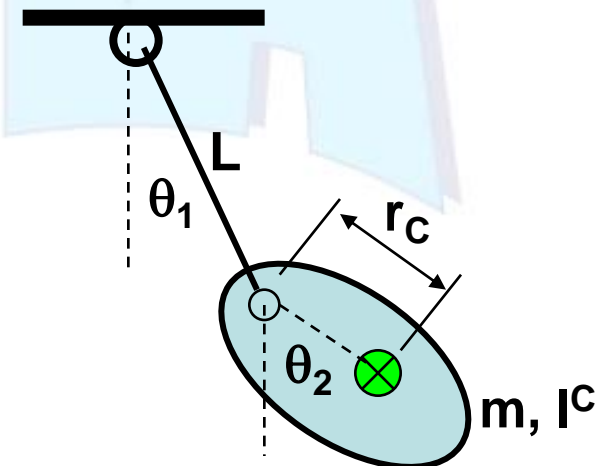
**Definition:** The **Generalized Force**  $\underline{Q} = \{Q_m\}$  is the set of coefficients of generalized virtual displacement  $\{\delta q^m\}$  in the expression of virtual work as:

$$\delta U = Q_m \delta q^m \quad (11.3)$$

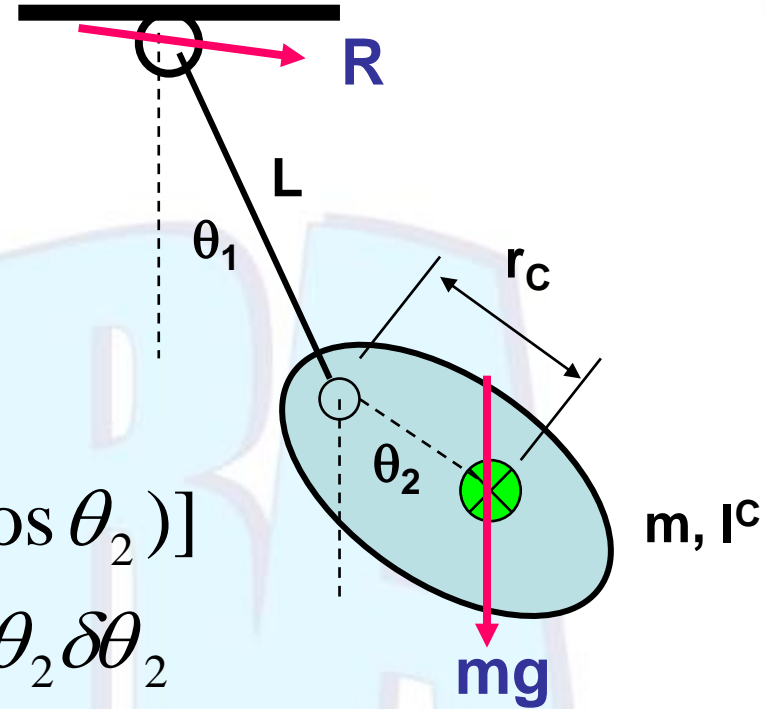
**Note that:** For a system of  $N$  particles, the working force in the expression of work is:

$$\delta U = \sum_{\beta=1}^N f_i^\beta \delta x_i^\beta \quad (11.4)$$

**Example:** Determine the Generalized Forces for the double pendulum shown?



Work done from equilibrium position to the existing position is:



$$U = -mg[L(1 - \cos \theta_1) + r_c (1 - \cos \theta_2)]$$

$$\delta U = -mgL \sin \theta_1 \delta \theta_1 - mgr_c \sin \theta_2 \delta \theta_2$$

Hence :

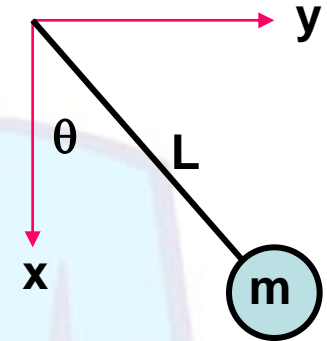
$$\text{Generalized Force} \equiv \underline{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -mgL \sin \theta_1 \\ -mgr_c \sin \theta_2 \end{Bmatrix}$$



**Some Useful Relations**: Functional relations between Cartesian coordinates and generalized coordinates;

$$1). \begin{cases} \{x_i\} \overset{f_i}{\leftrightarrow} \{q^m\} \\ x_i = f_i(q^m) \end{cases} \quad (11.5)$$

Ex:  $\begin{cases} x = L\cos\theta \\ y = L\sin\theta \end{cases}$



$q^m$  : *Generalized – Coordinates*

**2). By Chain Rule expansion of derivatives:**

$$\dot{x}_i = \frac{\partial x_i}{\partial q^m} \dot{q}^m = \frac{\partial x_i}{\partial q^1} \dot{q}^1 + \frac{\partial x_i}{\partial q^2} \dot{q}^2 + \dots \quad (11.6)$$



$$3). : \delta x_i = \frac{\partial x_i}{\partial q^m} \delta q^m \quad (11.7)$$

4). From relation (11.6), since  $\dot{x}_i$  is a linear function of  $\dot{q}^m$ , we have:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}^m} = \frac{\partial x_i}{\partial q^m} \quad (11.8)$$

5). By Chain Rule expansion of derivatives, we can write:

$$\frac{d}{dt} \left( \frac{\partial x_i}{\partial q^m} \right) = \frac{\partial \dot{x}_i}{\partial q^m} \quad (11.9)$$



5). By Chain Rule expansion of derivatives, we can write:

$$\frac{d}{dt} \left( \frac{\partial x_i}{\partial q^m} \right) = \frac{\partial \dot{x}_i}{\partial q^m} \quad (11.9)$$

**Proof:**

$$\frac{d}{dt} \left( \frac{\partial x_i}{\partial q^m} \right) = \frac{\partial}{\partial q^r} \left( \frac{\partial x_i}{\partial q^m} \right) \dot{q}^r = \frac{\partial^2 x_i}{\partial q^m \partial q^r} \dot{q}^r =$$

But  $q^m$  is independent of  $\dot{q}^r$ .

$$= \frac{\partial}{\partial q^m} \left( \frac{\partial x_i}{\partial q^r} \dot{q}^r \right) = \frac{\partial \dot{x}_i}{\partial q^m}$$

from Eq.(11.6) =  $\dot{x}_i$



## Lagrangian Equation of Motion

for

(Independent Set of Generalized Coordinates)

A mechanical system classified according to the coordinate description generally fall under one of the following categories:

**Rheonomic Systems:** The description of configuration by the generalized coordinates varies with time throughout the dynamic process.

$$x_i = f_i(q^m, t) \quad q^m = \text{Gen. Coords.} \quad (11.10)$$

**Scleronomic Systems:** The description is not explicitly time dependent.

$$x_i = f_i(q^m) \quad (11.11)$$



**Theorem-43:** Throughout the dynamic process of a mechanical system, the motion in the Independent Set of generalized coordinates  $\{q^m\}$  will satisfy the equation:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} = Q_m \quad m = 1, 2, \dots, N \quad (11.12)$$

**Proof:** Based on Leibniz Principle;  $\delta T - \delta U = 0$

**In  $\{x_i\}$  space:** Kinetic Energy expression for a single particle is;

$$T = \frac{1}{2} m \dot{x}_i \dot{x}_i \Rightarrow$$

$$\dot{T} = \frac{d}{dt} \left( \frac{1}{2} m \dot{x}_i \dot{x}_i \right) = m \ddot{x}_i \dot{x}_i = \frac{d}{dt} (m \dot{x}_i) \dot{x}_i = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) \dot{x}_i$$



$$\dot{T} = \frac{d}{dt} \left( \frac{1}{2} m \dot{x}_i \dot{x}_i \right) = m \ddot{x}_i \dot{x}_i = \frac{d}{dt} (m \dot{x}_i) \dot{x}_i = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) \dot{x}_i$$

$$dT = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) dx_i$$

$$\delta T = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) \delta x_i \quad \{\text{Virtual Change in K.E. in terms of Virtual Displacement}\} \quad (11.13)$$

**Going to  $\{q^m\}$  space:** Since by eq. (11.7);  $\delta x_i = \frac{\partial x_i}{\partial q^m} \delta q^m$

$$\delta T = \left[ \underbrace{\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right)}_U \underbrace{\frac{\partial x_i}{\partial q^m}}_V \right] \delta q^m$$

but :  $(dU)V = d(UV) - UdV$



$$\delta T = \left[ \frac{d}{dt} \underbrace{\left( \frac{\partial T}{\partial \dot{x}_i} \right)}_U \underbrace{\frac{\partial x_i}{\partial q^m}}_V \right] \delta q^m, \quad \text{but : } (dU)V = d(UV) - UdV$$

$$\delta T = \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \frac{\partial x_i}{\partial q^m} \right) - \frac{\partial T}{\partial \dot{x}_i} \frac{d}{dt} \left( \frac{\partial x_i}{\partial q^m} \right) \right] \delta q^m$$

**Eq.(11.8)** =  $\frac{\partial \dot{x}_i}{\partial \dot{q}^m}$ 
**Eq.(11.9)** =  $\frac{\partial \dot{x}_i}{\partial q^m}$

$$\delta T = \left[ \frac{d}{dt} \left( \frac{\partial T}{\cancel{\partial \dot{x}_i}} \frac{\cancel{\partial x_i}}{\partial \dot{q}^m} \right) - \frac{\partial T}{\cancel{\partial \dot{x}_i}} \frac{\cancel{\partial x_i}}{\partial q^m} \right] \delta q^m$$

$$\delta T = \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} \right] \delta q^m \quad (11.14)$$



$$\delta T = \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} \right] \delta q^m \quad (11.14)$$

$$\text{But : } \begin{cases} \delta U = \delta T & (\text{From Leibniz Principle}) \\ \delta U = Q_m \delta q^m & (\text{Def. of Generalized Force}) \end{cases} \quad (11.3)$$

From Equations (11.14) and (11.3) we have:

$$\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} - Q_m \right] \delta q^m = 0 \quad (11.15)$$

Since  $\delta q^m$  is independent and arbitrary, then:

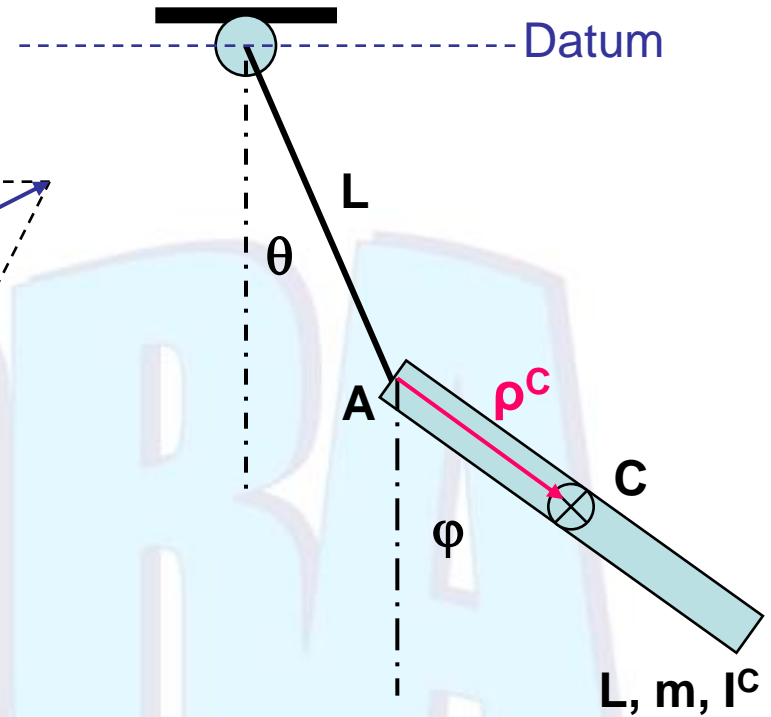
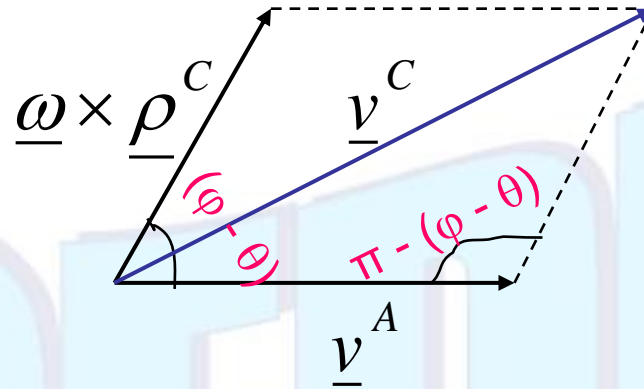
$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} = Q_m \quad m = 1, 2, \dots, N \quad (11.12)$$

Lagrangian Equation of Motion for Independent Set of Generalized Coordinates. Applies to both **Scleronomic** and **Rheonomic** Systems.



**Example:** Express Differential Equations of Motion of the following system:

**1. Motion:**



$$\underline{v}^C = \underline{v}^A + \underline{\omega} \times \underline{\rho}^C, \quad \omega = \dot{\phi}$$

$$(v^C)^2 = (\ell \dot{\theta})^2 + \left(\frac{\ell}{2} \dot{\phi}\right)^2 - 2(\ell \dot{\theta})\left(\frac{\ell}{2} \dot{\phi}\right) \cos[\pi - (\phi - \theta)] =$$

$$= (\ell^2 \dot{\theta}^2) + \frac{1}{4} \ell^2 \dot{\phi}^2 + \ell^2 \dot{\theta} \dot{\phi} \cos(\phi - \theta)$$

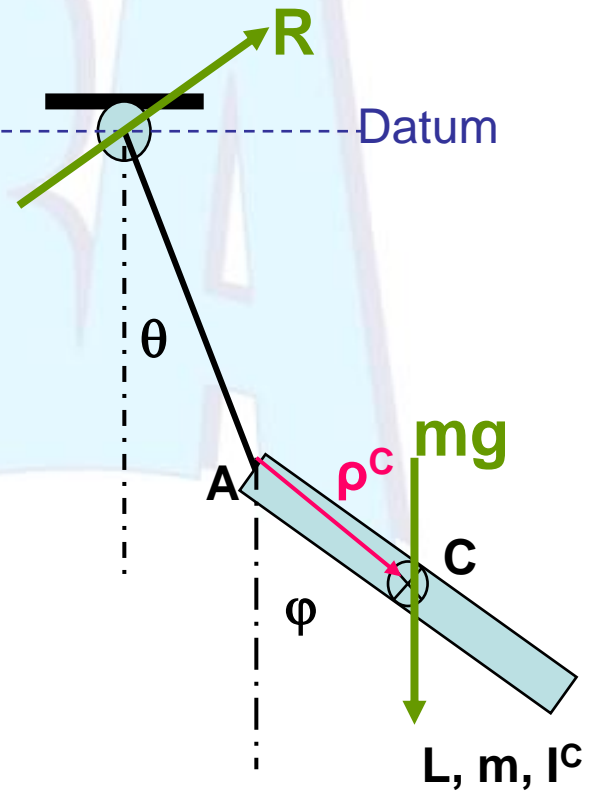


## 2. Kinetic Energy:

$$T = T^{(e)} + T^{(i)} = \frac{1}{2} m v_C^2 + \frac{1}{2} I^C \omega^2$$

$$T = \frac{1}{2} m [\ell^2 \dot{\theta}^2 + \frac{1}{4} \ell^2 \dot{\varphi}^2 + \ell^2 \dot{\theta} \dot{\varphi} \cos(\varphi - \theta)] + \frac{1}{2} \left( \frac{m \ell^2}{12} \right) \dot{\varphi}^2$$

$$T = \frac{1}{2} m [\ell^2 \dot{\theta}^2 + \frac{1}{3} \ell^2 \dot{\varphi}^2 + \ell^2 \dot{\theta} \dot{\varphi} \cos(\varphi - \theta)]$$



## 3. Virtual Work: Free Body Diagram

$$U = mg \left[ \ell \cos \theta + \frac{\ell}{2} \cos \varphi \right]$$

$$\delta U = mgl \left[ -\sin \theta \delta \theta - \frac{1}{2} \sin \varphi \delta \varphi \right] \equiv Q_m \delta q^m$$

$$\{Q_m\} = \begin{Bmatrix} Q_\theta \\ Q_\varphi \end{Bmatrix} = \begin{Bmatrix} -mgl \sin \theta \\ -\frac{1}{2} mgl \sin \varphi \end{Bmatrix}$$



#### 4. Apply the Lagrangian Equation of Motion:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}^m} \right) - \frac{\partial T}{\partial q^m} = Q_m ; \quad m = 1, 2$$

For  $q^1 = \theta \Rightarrow$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2 \dot{\theta} + \frac{1}{2} ml^2 \dot{\phi} \cos(\phi - \theta)$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = ml^2 \ddot{\theta} + \frac{1}{2} ml^2 \ddot{\phi} \cos(\phi - \theta) + \frac{1}{2} ml^2 \dot{\phi} (\dot{\phi} - \dot{\theta}) [-\sin(\phi - \theta)]$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2} ml^2 \dot{\theta} \dot{\phi} \sin(\phi - \theta)$$

**Substituting into the Lagrange's equation, we get:**

$$ml^2 \ddot{\theta} + \frac{1}{2} ml^2 \ddot{\phi} \cos(\phi - \theta) + \frac{1}{2} ml^2 \dot{\phi}^2 \sin(\phi - \theta) = -mgl \sin \theta \quad \leftarrow$$

For  $q^2 = \phi$ , we can similarly solve for the second equation of motion.



**Example:** Express Differential Equation of Motion of the YOYO shown (an example for *Rheonomic* System where  $x_i = f_i(q^m, t)$ )?

Consider the Pendulum as a 1-DOF system described by “ $\theta$ ”. Then “ $\underline{F}$ ” (tensile force) does no virtual work, because “ $L$ ” does not change in virtual displacement, which holds time constant.

$$L = L_0 - vt \quad \Rightarrow \quad \dot{L} = -v \quad \Rightarrow \quad \delta L = -v \delta t = 0$$

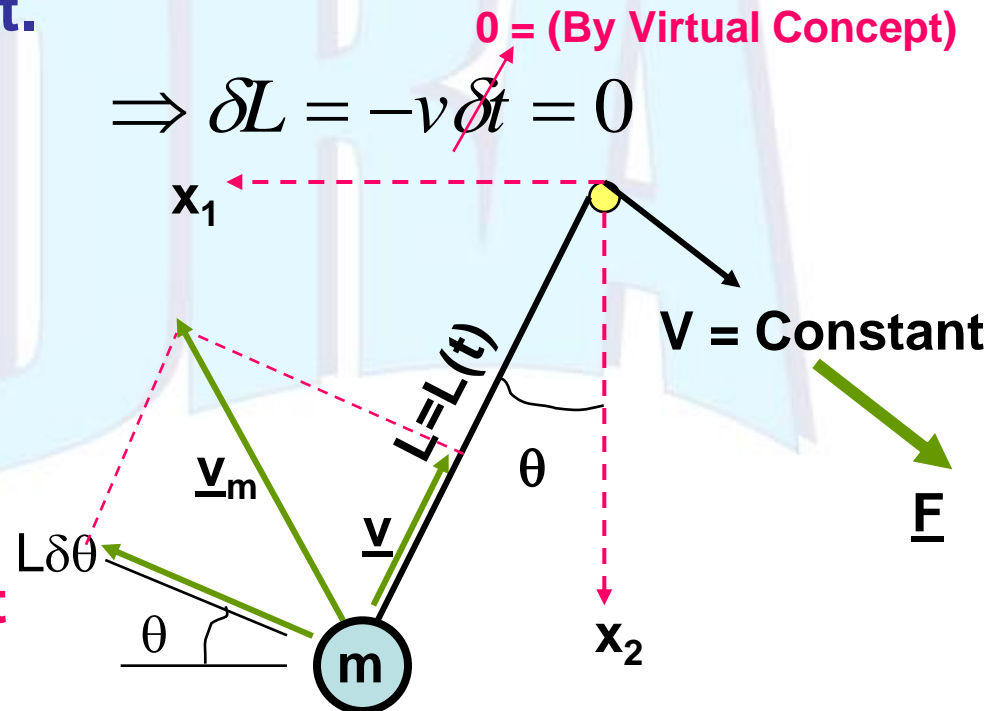
$$\delta U = F \cdot \delta L = 0$$

### 1. Motion:

Initial Length is;  $L_0$

Current Length is;  $L = L_0 - vt$

Current Position is;



$$\underline{r} = L \sin \theta \underline{e}_1 + L \cos \theta \underline{e}_2 =$$

$$= \underbrace{(L_0 - vt)}_{x_1} \sin \theta \underline{e}_1 + \underbrace{(L_0 - vt)}_{x_2} \cos \theta \underline{e}_2$$

**Note that:**  $x_i = f_i(\theta, t)$

$$\underline{\dot{r}} = L \dot{\theta} \cos \theta \underline{e}_1 - L \dot{\theta} \sin \theta \underline{e}_2 - v \sin \theta \underline{e}_1 - v \cos \theta \underline{e}_2$$

$$(v_m)^2 = L^2 \dot{\theta}^2 + v^2$$

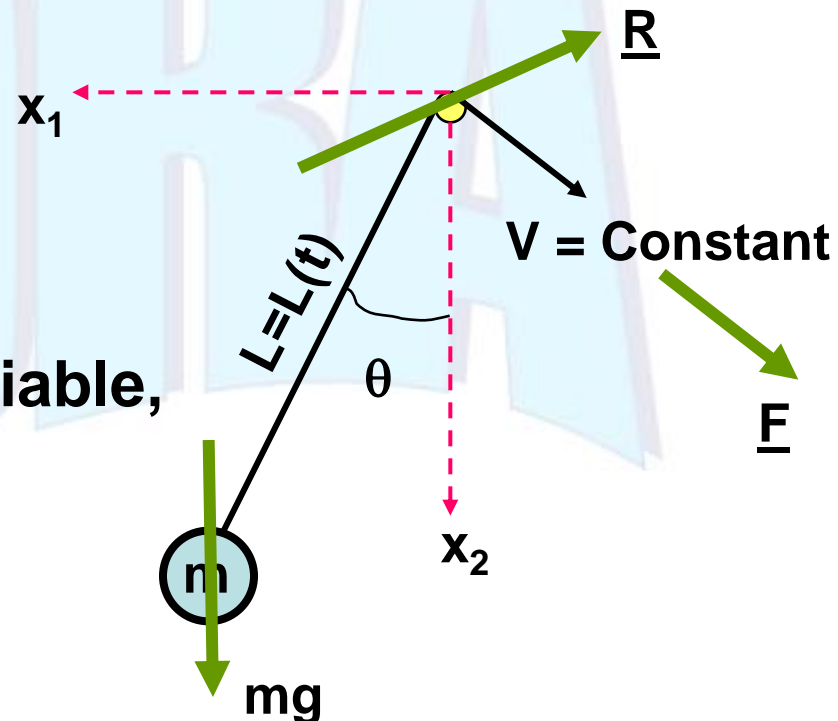
## 2. Kinetic Energy:

$$T = \frac{1}{2} m v_m^2 = \frac{1}{2} m (L^2 \dot{\theta}^2 + v^2)$$

## 3. Virtual Work: $\theta$ is the only variable, and Free Body Diagram;

$$\delta U = -mgL \sin \theta \delta \theta \Rightarrow$$

$$Q_\theta = -mgL \sin \theta$$



## 4. Apply Lagrange's Equation of Motion:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = Q_{\theta}$$

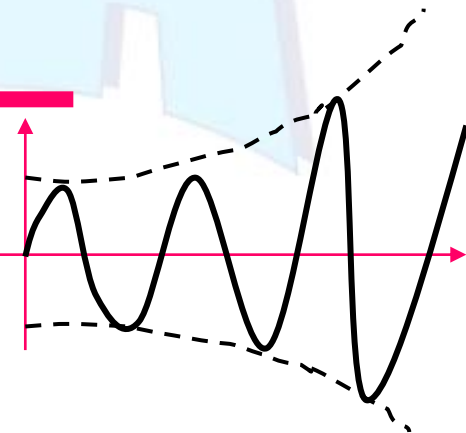
$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = \frac{d}{dt} (mL^2 \dot{\theta}) = mL^2 \ddot{\theta} + 2mL\dot{L}\dot{\theta} = mL^2 \ddot{\theta} - 2mLv\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0$$

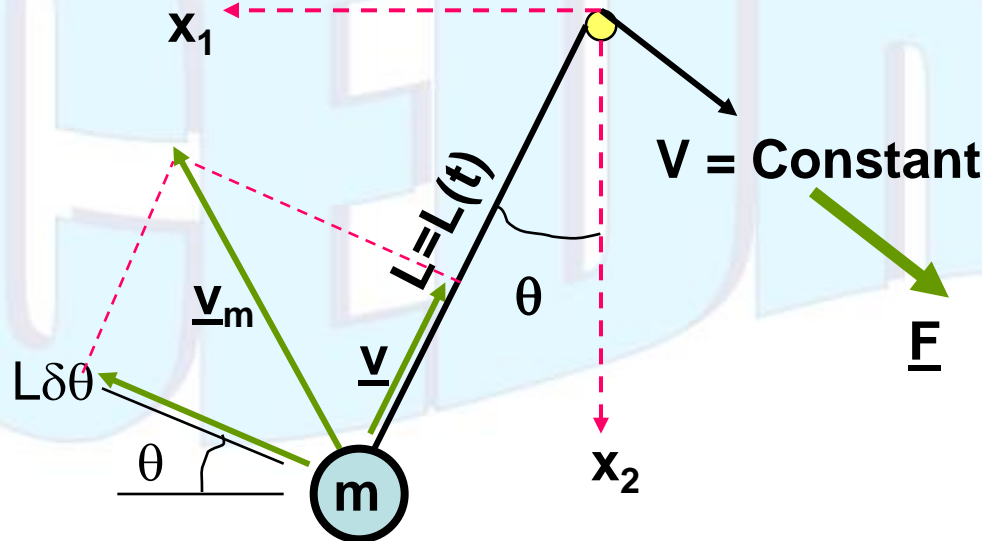
$$mL^2 \ddot{\theta} - 2mLv\dot{\theta} + mgL \sin \theta = 0$$

$$\ddot{\theta} - \frac{2v}{L} \dot{\theta} + \frac{g}{L} \theta = 0 \quad (\text{For Small Angles})$$

(Negative damping results in an Unstable system)



**Another Approach to the YOYO Problem:** “ $\theta$ ” and “ $L$ ” are constrained generalized coordinates that satisfy  $L=L(t)$ . Then, “ $L$ ” increases by “ $\delta L$ ” in a virtual movement, and “ $\underline{F}$ ” does virtual work. Then, we need to formulate the problem by **Lagrange’s Equation for Dependent Set of Generalized Coordinates.**



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