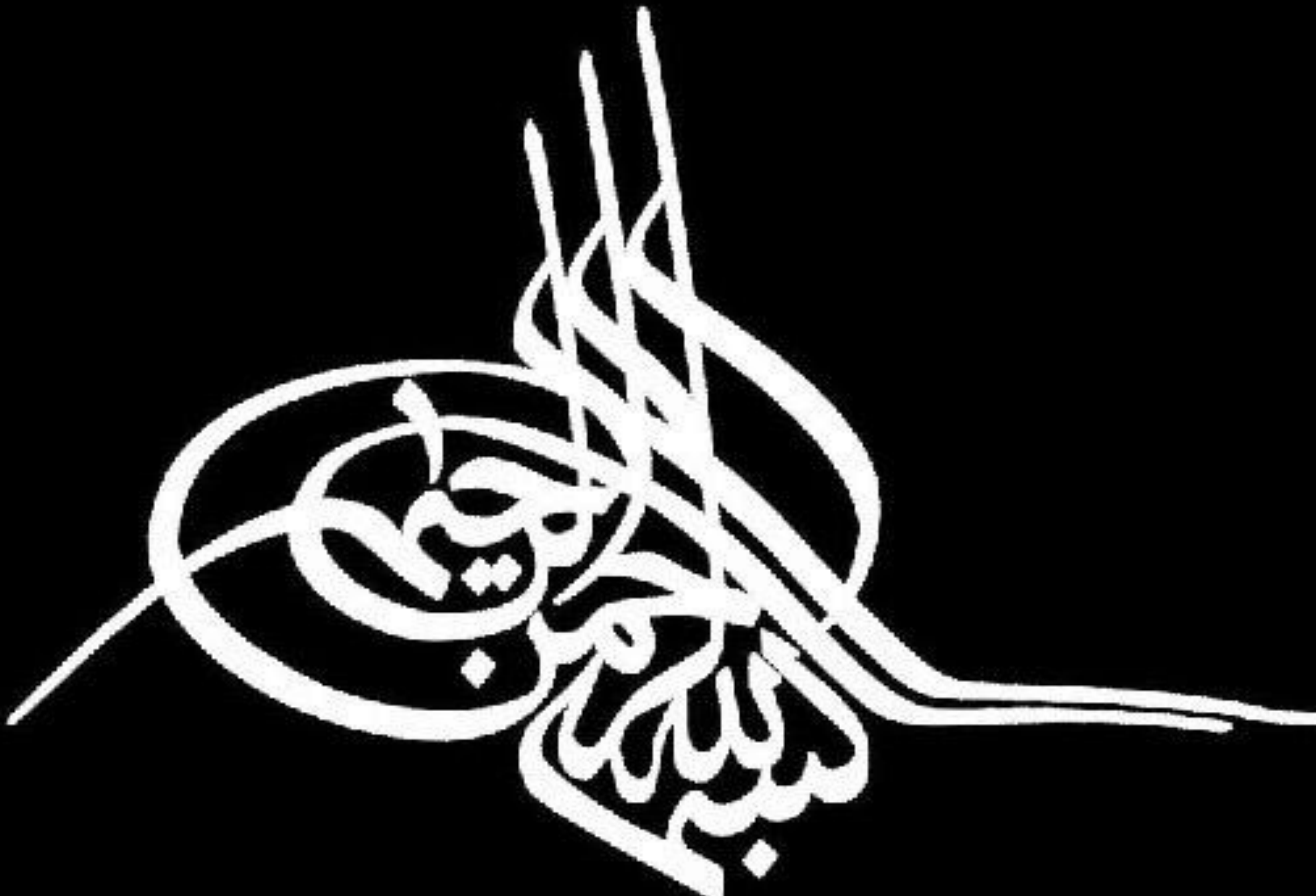

بسم الله الرحمن الرحيم

Lagrangian Equation of Motion

Purpose:

- **To extend the *Energy* approach in deriving equations of motion (i.e. Lagrange's Method) for Mechanical Systems.**

Topics:

- **Generalized Coordinates**
- **Lagrangian Equation of Motion for *Independent* Set of Generalized Coordinates**
- **Lagrangian Equation of Motion for *Dependent* Set of Generalized Coordinates**
- **Hamiltonian Principle**

Lagrangian Equation of Motion
for
(*Dependent Set of Generalized Coordinates*)

When functional relations exist among the generalized coordinates, the Lagrange's Equation must include the Constraint Relations.

***Definition*: A Constrained Generalized Coordinate Set has the generalized coordinates related by a system of functions, $g_r(q^m)=0$, called Constraints.**

$$g_r(q^1,\ldots,q^N)=0,\quad r=1,\ldots,R \qquad \textbf{(11.16)}$$

***Degrees-of-Freedom* =**

# of Dependent Generalized Coordinates - # of Constraints

Constraints are occasionally represented in differential forms (i.e. most common is the functional relationship among generalized velocities).

***Concept of Constraint*:**

(a) Integral Expression (Integrated form of constraint; *Holonomic* Constraint):

$$g_r(q^m, t) = 0, \quad r = 1, \ldots, R; \quad m = 1, \ldots, N \qquad \textbf{(11.17)}$$

(b) Differential Expression (Differentiated form of constraint; *Non-Holonomic* Constraint)

$$\dot{g}_r = \frac{\partial g_r}{\partial q^m}\dot{q}^m + \frac{\partial g_r}{\partial t} = 0, \quad r = 1, \ldots, R < N; \quad m = 1, \ldots, N$$

$$C_{rm}\dot{q}^m + C_r = 0 \quad \Rightarrow \quad C_{rm}dq^m + C_r dt = 0 \qquad \textbf{(11.18)}$$

$$\{C_{rm}\} = \frac{\partial g_r}{\partial q^m}$$ **(*Constraint Coefficients*; not constant and function of q^m)**

In Virtual Form:

$$\delta g_r = \frac{\partial g_r}{\partial q^m}\delta q^m + \frac{\partial g_r}{\partial t}\cancel{\delta t} = 0$$

(when δt is set to be zero by virtual concept)

$$\delta g_r = C_{rm}\delta q^m = 0 \qquad \textbf{(11.19)}$$

Equation (11.19) is an *Integrated* or *Differential* form of constraint expressed by virtual displacement.

If it is possible to go from (11.19) to (11.17), then the system is *Holonomic*. (Greek word meaning Integrable)

If it is not possible to go from (11.19) to (11.17), then the system is *Non-Holonomic*.

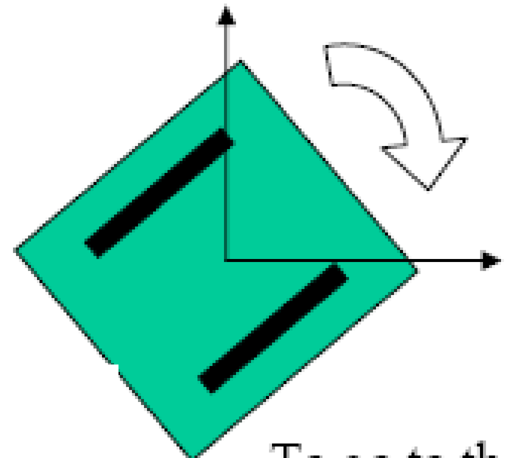
Definition **(Holonomic System): When the constraints of a system are in the integrated (integrable differentia) form, the number of Generalized Coordinates can be reduced to correspond to the Degrees-of-Freedom of the system.**

Definition **(Non-Holonomic System): When some of the constraints of the system are in the non-integrable (differential) form, the number of Generalized Coordinates can not be reduced to correspond to the Degrees-of-Freedom of the system.**

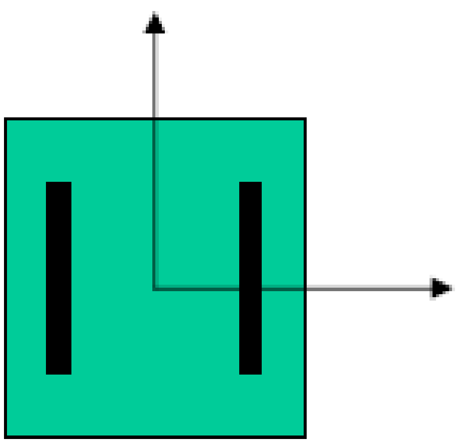
Equation (11.19): $C_{rm}\delta q^{m} = 0$ **implies that {δq^m} is no longer an independent set.**

Example: Non-Holonomic Constraints often arise in systems having parts that roll.

Example: (Non-Holonomic Constraint) ***A constraint on velocity does not induce a constraint on position.*** **For a wheeled robot, it can instantaneously move in some directions (forwards and backwards), but not others (side to side).**



To go to the right, the robot must first turn, and then drive forward



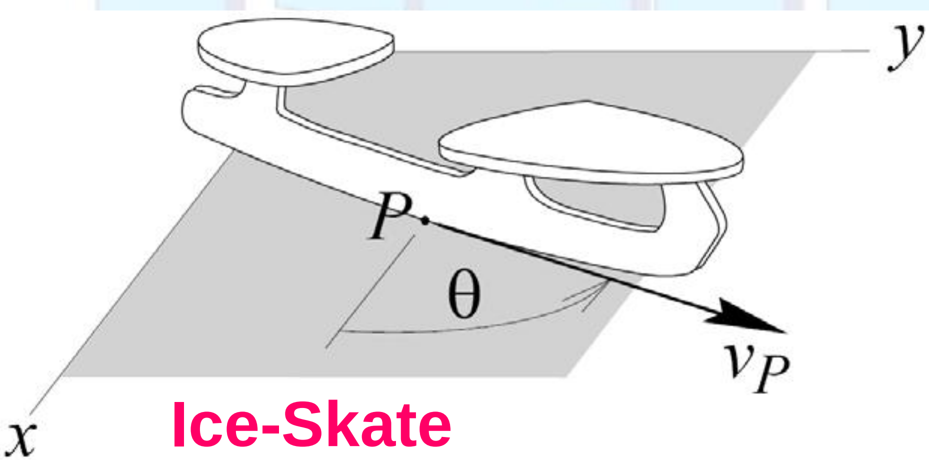
The robot can instantly move forward and back, but can not move to the right or left without the wheels slipping.

Mathematical Description of the Constraint: For a differential drive, this is:

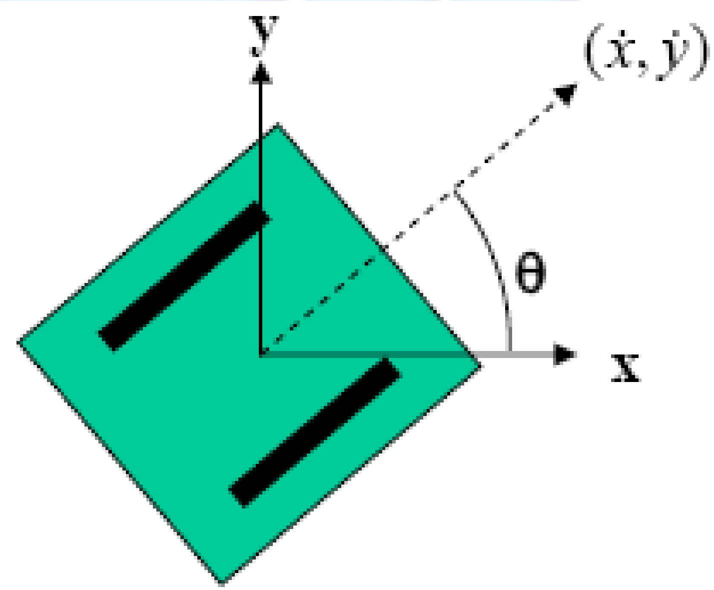
$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0$$

The directions car can't move are:

1. **If θ = 0, then velocity in y = 0, and**
2. **If θ = 90, then velocity in x = 0.**



Ice-Skate



Example: (Holonomic System)

***A Person Walking*; you can instantly step to the left and right, as well as going forward and backward. In other words, your velocity in plane is not restricted.**

An Omni wheeled robot; is another example of Holonomic System. It can roll forwards and backwards, as well as sideways.

Recall: since δq^m is no longer an independent set, then its coefficient in equation (11.15) can not be set to zero. Therefore, the Lagrange's Equation in the above form can not be applied to *Constraint (Dependent) form of Generalized Coordinates*.

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^m}) - \frac{\partial T}{\partial q^m} - Q_m]\delta q^m = 0 \qquad \textbf{(11.15)}$$

***Theorem-44*: The dynamic process of a mechanical system, described in a set of constrained generalized coordinates, must satisfy the following set of equations:**

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^m}) - \frac{\partial T}{\partial q^m} + C_{rm}\lambda^r = Q_m \qquad \begin{cases} m = 1,\ldots,N \\ r = 1,\ldots,R \end{cases} \qquad \textbf{(11.20)}$$

Where: λ^r are R-number of Lagrangian Multipliers corresponding to the number of constraints.

Proof: Lets add the null combination of *Lagrangian Multipliers* and the constraints to Equation (11.15) as:

$$\lambda^r C_{rm} \delta q^m = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^m}\right) - \frac{\partial T}{\partial q^m} + \lambda^r C_{rm} - Q_m\right] \delta q^m = 0 \qquad \begin{cases} m = 1, \ldots, N \\ r = 1, \ldots, R \end{cases}$$

(11.21)

Let us now separate Equation (11.21) into two groups:

1. **The linear combination of those dependent variables $\{\delta q^1, \ldots, \delta q^R\}$, and**
2. **The linear combination of those independent set $\{\delta q^{R+1}, \ldots, \delta q^N\}$.**

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^s}) - \frac{\partial T}{\partial q^s} + \lambda^r C_{rs} - Q_s]\delta q^s +$$

$$+[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^k}) - \frac{\partial T}{\partial q^k} + \lambda^r C_{rk} - Q_k]\delta q^k = 0 \qquad ;\begin{cases} s = 1,\ldots,R \\ k = R+1,\ldots,N \end{cases}$$

(11.22)

Now, we can impose to choose "R" number of "λ's" so that R equations become zero. In other words, a set of {λ^r} can be found such that the coefficients of {δq^s} in equation (11.22) become zero.

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^s}) - \frac{\partial T}{\partial q^s} + \lambda^r C_{rs} = Q_s \qquad \textbf{(11.23)}$$

(For a special set of {λ^r} with dependent variables)

What remains from Equation (11.22), are the second group, as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{k}}\right)-\frac{\partial T}{\partial q^{k}}+\lambda^{r} C_{rk}=Q_{k} \qquad \textbf{(11.24)}$$

(For an independent $\{\delta q^k\}$)

Equations (11.23) and (11.24) together form a complete set for Equation (11.20).

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{m}}\right)-\frac{\partial T}{\partial q^{m}}+C_{rm}\lambda^{r}=Q_{m} \qquad \begin{cases} m=1,\ldots,N \\ r=1,\ldots,R \end{cases}$$

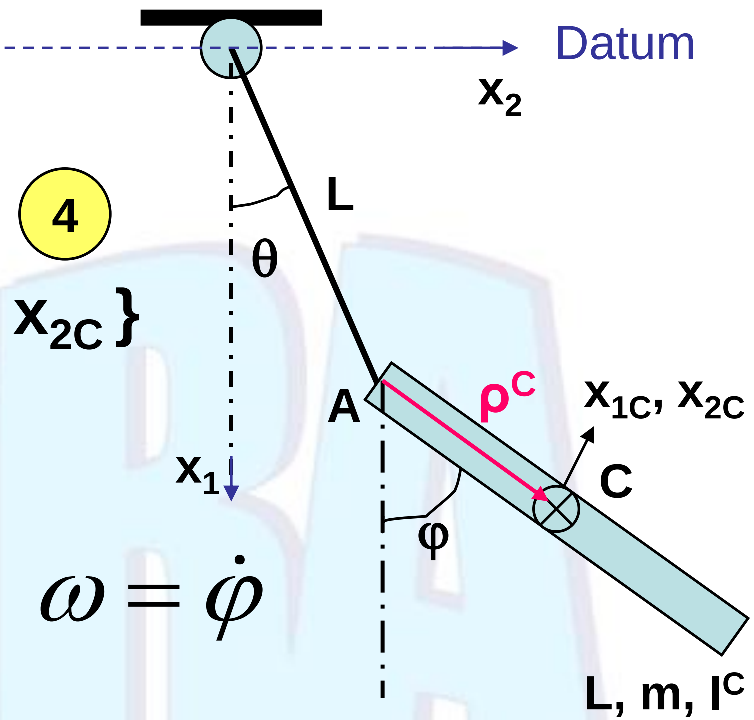
(11.20)

Conceptually, the *Lagrangian Multipliers* help to reduce the dependent N-set of generalized coordinates into the Independent (N-R) set.

In other words, the introduction of the Lagrangian Multipliers expands the space to (N+R) for $\{q^m, \lambda^r\}$ without constraints.

(N+R) = set of equations

Example: Express Differential Equations of Motion of the following system:



1. Motion:

1 2 3 4

Let's choose $\{q^m\} = \{\ \theta\ ,\ \varphi\ ,\ x_{1C}\ ,\ x_{2C}\ \}$

$$(v^C)^2 = (\dot{x}_{1C})^2 + (\dot{x}_{2C})^2, \quad \omega = \dot{\varphi}$$

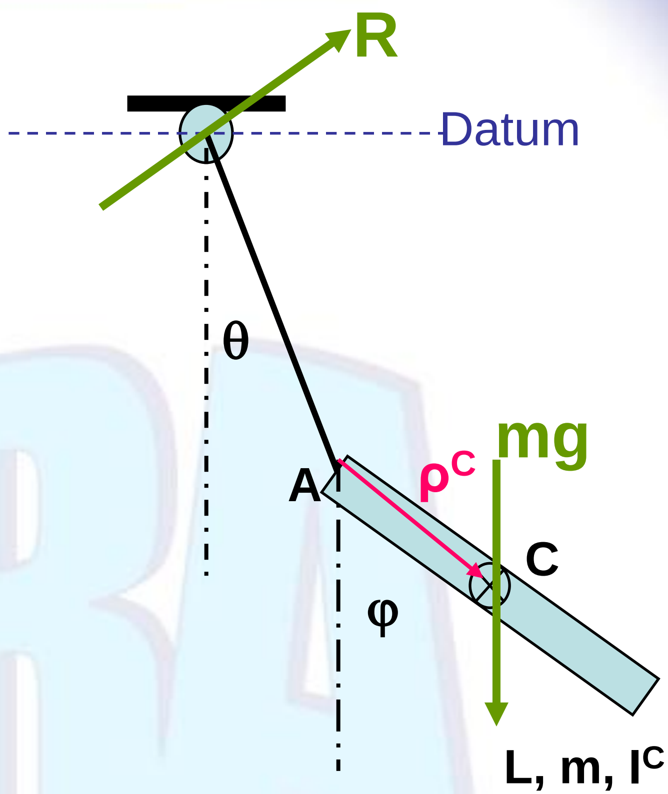
2. Kinetic Energy:

$$T = \frac{1}{2}m[(\dot{x}_{1C})^2 + (\dot{x}_{2C})^2] + \frac{1}{2}I^C\dot{\varphi}^2$$

3. Virtual Work: Free Body Diagram

$$\delta U = mg\,\delta x_{1C}$$

$$\{Q_m\} = \begin{Bmatrix} 0 \\ 0 \\ mg \\ 0 \end{Bmatrix}$$



4. Constraints:

$$g_1 = x_{1C} - L\cos\theta - \rho^C \cos\varphi = 0 \qquad (1)$$

$$g_2 = x_{2C} - L\sin\theta - \rho^C \sin\varphi = 0 \qquad (2)$$

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$$g_1 = x_{1C} - L\cos\theta - \rho^C \cos\varphi = 0 \qquad (1)$$

$$g_2 = x_{2C} - L\sin\theta - \rho^C \sin\varphi = 0 \qquad (2)$$

$$C_{rm} = \frac{\partial g_r}{\partial q^m}$$

C_{rm}	m \ r	θ	φ	x_{1C}	x_{2C}
	1	$L\sin\theta$	$\rho^C\sin\varphi$	1	0
	2	$-L\cos\theta$	$-\rho^C\cos\varphi$	0	1

5. Lagrangian Equation of Motion:

5.1 For Coordinate "$q^1 = \theta$";

$$\frac{d}{dt}\left(\cancelto{0}{\frac{\partial T}{\partial \dot{\theta}}}\right) - \cancelto{0}{\frac{\partial T}{\partial \theta}} + \lambda^1 C_{1\theta} + \lambda^2 C_{2\theta} = Q_\theta$$

$$\lambda^1 (L \sin\theta) + \lambda^2 (-L\cos\theta) = 0 \qquad \textbf{(3)}$$

5.2 For Coordinate "$q^2 = \varphi$";

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} + \lambda^1 C_{1\varphi} + \lambda^2 C_{2\varphi} = Q_\varphi$$

$$I^C \ddot{\varphi} + \lambda^1 (\rho^C \sin\varphi) + \lambda^2 (-\rho^C \cos\varphi) = 0 \qquad \textbf{(4)}$$

5.3 For Coordinate "$q^3 = x_{1C}$";

$$m\ddot{x}_{1C} + \lambda^1(1) = mg \qquad (5)$$

5.4 For Coordinate "$q^4 = x_{2C}$";

$$m\ddot{x}_{2C} + \lambda^2(1) = 0 \qquad (6)$$

We are faced with six equations and six unknowns, that can be solved for {θ, φ, x_{1C} , x_{2C} , λ^1 , λ^2 }.

Constraint equations and their derivatives are used and substituted in equations (5) and (6) till x_{1C} , x_{2C} , λ^1 , λ^2 are eliminated from all equations, and two independent equations in terms θ and φ and their derivatives are remained.

متشکرم