
بسم الله الرحمن الرحيم

Lagrangian Equation of Motion

***Purpose*:**

- **To extend the *Energy* approach in deriving equations of motion (i.e. Lagrange's Method) for Mechanical Systems.**

***Topics*:**

- **Generalized Coordinates**
- **Lagrangian Equation of Motion for *Independent* Set of Generalized Coordinates**
- **Lagrangian Equation of Motion for *Dependent* Set of Generalized Coordinates**
- **Hamiltonian Principle**

Hamilton’s Principles

It is an integral principle and considers the configuration of a system between the time interval (t_0 , t_1).

Advantages; **“*Dynamics Formulation*” is:**

1. **Reduced to the evaluation of a scalar definite integral,**
2. **Coordinate system independent in expressing the integrand.**

Let us consider a system of N particles. Using *D'Alemberts's Principle* and the *Principle of Virtual Work* we have:

$$\delta U = \sum_{\beta=1}^{N} [f_i^{\beta} - \frac{d}{dt}(m_\beta \dot{x}_i^{\beta})] \cdot \delta x_i^{\beta} = 0 \qquad or$$

$$\delta U = \sum_{\beta=1}^{N} [\underline{f}^{\beta} - \underbrace{\frac{d}{dt}(m_\beta \dot{\underline{r}}^{\beta})] \cdot \delta \underline{r}^{\beta}} = 0 \qquad \textbf{(11.21)}$$

Can be written as:

(11.22)

$$\sum_{\beta=1}^{N} \frac{d}{dt}(m_\beta \dot{\underline{r}}^{\beta}) \cdot \delta \underline{r}^{\beta} = \sum_{\beta=1}^{N} \frac{d}{dt}(m_\beta \dot{\underline{r}}^{\beta} \cdot \delta \underline{r}^{\beta}) - \sum_{\beta=1}^{N} m_\beta \dot{\underline{r}}^{\beta} \cdot \delta \dot{\underline{r}}^{\beta}$$

Recall that the Kinetic Energy for a System of Particles is:

$$T = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \underline{\dot{r}}^{\beta} \cdot \underline{\dot{r}}^{\beta}$$

$$\delta T = \sum_{\beta=1}^{N} m_\beta \underline{\dot{r}}^{\beta} \cdot \delta \underline{\dot{r}}^{\beta}$$

Variation in Kinetic Energy

Substitute in equation (11.22)

$$\sum_{\beta=1}^{N} \frac{d}{dt}(m_\beta \underline{\dot{r}}^{\beta}) \cdot \delta \underline{r}^{\beta} = \sum_{\beta=1}^{N} \frac{d}{dt}(m_\beta \underline{\dot{r}}^{\beta} \cdot \delta \underline{r}^{\beta}) - \delta T$$

(11.23)

On the other hand, since Virtual Work is defined as:

$$\delta U = \sum_{\beta=1}^{N} \underline{f}_{\beta} \cdot \delta \underline{r}^{\beta} \qquad \textbf{(11.24)}$$

Substituting Equations (11.23) and (11.24) into equation (11.21), we obtain:

$$\delta U + \delta T = \sum_{\beta=1}^{N} \frac{d}{dt}(m_{\beta} \dot{\underline{r}}^{\beta} \cdot \delta \underline{r}^{\beta}) \qquad \textbf{(11.25)}$$

Integrating Equations (11.25) over the time interval t_0 to t_1 results in:

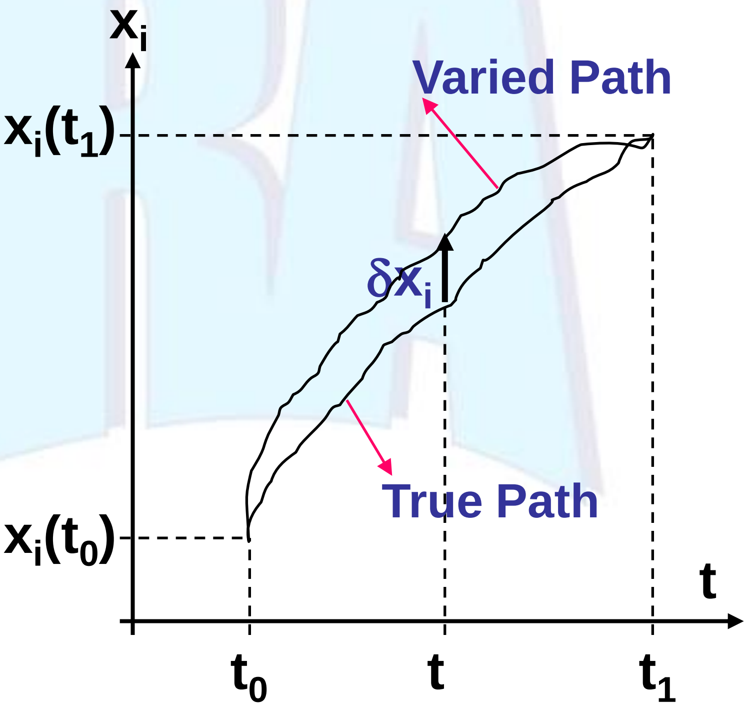
$$\int_{t_0}^{t_1} (\delta U + \delta T)dt = [\sum_{\beta=1}^{N} m_{\beta} \dot{\underline{r}} \cdot \delta \underline{r}^{\beta}]_{t_0}^{t_1} \qquad \textbf{(11.26)}$$

$$but \quad \delta \underline{r}^{\beta}(t_0) = \delta \underline{r}^{\beta}(t_1) = 0$$

$$\int_{t_0}^{t_1} (\delta U + \delta T)\,dt = \Big[\sum_{\beta=1}^{N} m_\beta \dot{\underline{r}} \cdot \delta \underline{r}^\beta \Big]_{t_0}^{t_1} \qquad \textbf{(11.26)}$$

$$but \quad \delta \underline{r}^\beta (t_0) = \delta \underline{r}^\beta (t_1) = 0$$

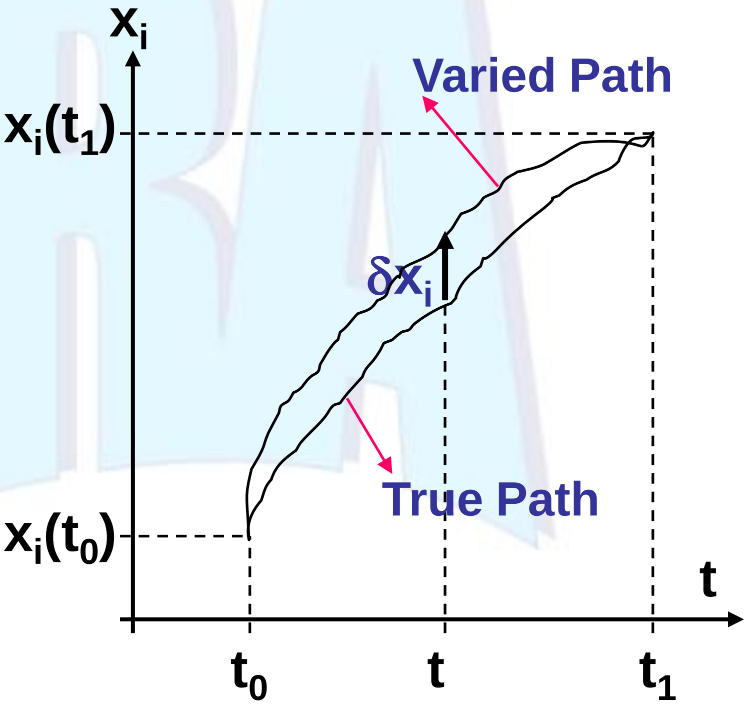
$$\int_{t_0}^{t_1} (\delta U + \delta T)\,dt = 0 \qquad \textbf{(11.27)}$$



General Form of
Hamilton's Principle

$$\int_{t_0}^{t_1} (\delta U + \delta T)dt = 0 \qquad (11.27)$$

General Form of *Hamilton's Principle*: It states that "the true path followed by the dynamic system to go from $\underline{r}(t_0)$ to $\underline{r}(t_1)$ is such that the time integral of the sum of the virtual kinetic energy change and virtual work vanishes when subjected to virtual displacements from the true path". Hamilton's Principle can be applied to both Non-Holonomic and Non-Conservative systems.



Special Cases: **when forces are conservative and the virtual work is related to the change in potential energy V by $\delta U = -\delta V$, we have:**

$$L = T - V \equiv Lagrangian(a \quad scalar \quad function)$$

$$where: T = T(q_m, \dot{q}_m, t) \quad , and \quad V = V(q_m, t)$$

$$Then; \quad Equation\,(11.27) \quad becomes:$$

$$\int_{t_0}^{t_1} \delta L dt = 0 = \int_{t_0}^{t_1} \delta(T - V) dt \qquad \textbf{(11.28)}$$

If the system is Holonomic, then equation (11.28) becomes:

$$\delta I = \delta \int_{t_0}^{t_1} L dt = 0 \Rightarrow \quad I = \int_{t_0}^{t_1} L dt \qquad \textbf{(11.29)}$$

$$\delta I = \delta \int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \quad I = \int_{t_0}^{t_1} L dt \qquad \textbf{(11.29)}$$

Equation (11.29) states that the true path followed by a conservative holonomic system to go from $\underline{r}(t_0)$ to $\underline{r}(t_1)$ is such that the time integral " I " is *extremized*.

***Proof of Lagrange's Equation from Hamilton's Principle*:**

$$\underline{\textbf{Hamilton's Principle}} \begin{cases} \int_{t_0}^{t_1} (\delta U + \delta T) dt = 0 \\ \delta \underline{r}(t_0) = \delta \underline{r}(t_1) = \underline{0} \end{cases} \qquad \textbf{(11.27)}$$

For Holonomic System of N-Particles with m degrees of freedom we have:

$q_1, q_2, \ldots, q_m$ = Generalized Coordinates = $\{q_m\}$ Space
$\underline{r} = \underline{r}(q_1, q_2, \ldots, q_m, t)$ = Vector Coordinates of Particles

Then, the Total Kinetic Energy for the system is:

$$T = \frac{1}{2}\sum_{\beta=1}^{N} m_\beta \dot{\underline{r}}^\beta \cdot \dot{\underline{r}}^\beta$$

$$T = T(q_1, q_2, \ldots, q_m, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_m, t) \qquad \textbf{(11.30)}$$

But, virtual work done by generalized forces are:

$$\left\{ \begin{array}{l} \delta U = \sum_m Q_m \delta q_m \quad \textbf{substitute in (11.27)} \\ \int_{t_0}^{t_1} (\delta T + \sum_m Q_m \delta q_m) dt = 0 \end{array} \right. \qquad \textbf{(11.31)}$$

Taking the variation of T using equation (11.30), and noting that " $\delta t=0$ ", we have:

$$\delta T = \sum_m \frac{\partial T}{\partial q_m}\delta q_m + \sum_m \frac{\partial T}{\partial \dot{q}_m}\delta \dot{q}_m$$

substitute in (11.31)

$$\int_{t_0}^{t_1}\sum_m \left[\left(\frac{\partial T}{\partial q_m} + Q_m\right)\delta q_m + \frac{\partial T}{\partial \dot{q}_m}\delta \dot{q}_m\right]dt = 0 \qquad \textbf{(11.32)}$$

Integrating the last term of Eq. (11.32) by parts, we have:

$$\int_{t_0}^{t_1}\frac{\partial T}{\partial \dot{q}_m}\delta \dot{q}_m\,dt = \cancel{\left[\sum_m \frac{\partial T}{\partial \dot{q}_m}\delta q_m\right]_{t_0}^{t_1}} - \int_{t_0}^{t_1}\sum_m \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right)\delta q_m\,dt$$

0 $[\delta q_m(t_0) = \delta q_m(t_1) = 0]$

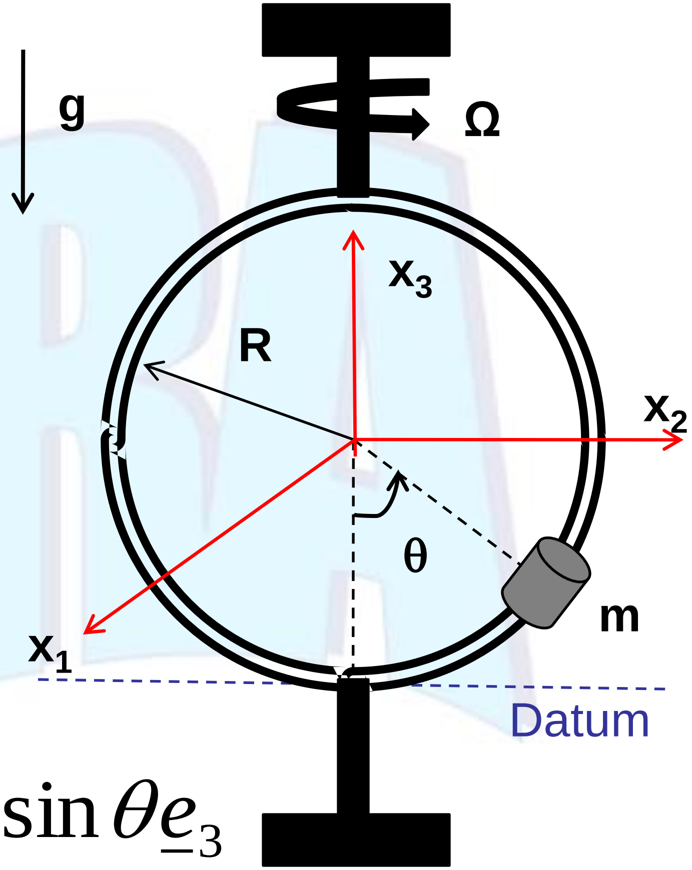
substituting in equation (11.32), we have:

$$\int_{t_0}^{t_1} \sum_m \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right) + \frac{\partial T}{\partial q_m} + Q_m \right] \delta q_m \, dt = 0 \qquad \textbf{(11.33)}$$

Since in Holonomic Systems, the generalized coordinates form an independent set, therefore, the coefficients of each δq_m in equation (11.33) must be zero. Therefore:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right) - \frac{\partial T}{\partial q_m} = Q_m \qquad m = 1,2,\ldots,M \qquad \textbf{(11.34)}$$

Example: A bead of mass m is free to slide on a hoop of radius R as shown. The hoop is rotating with the constant angular velocity Ω. Find the equation of motion using Hamilton's principle?



1. Motion:

Let x_1, x_2, x_3 be attached to the hoop.

$$\underline{r} = R\sin\theta\,\underline{e}_2 - R\cos\theta\,\underline{e}_3$$

$$\underline{v} = -R\Omega\sin\theta\,\underline{e}_1 + R\dot{\theta}\cos\theta\,\underline{e}_2 + R\dot{\theta}\sin\theta\,\underline{e}_3$$

2. Kinetic Energy:

$$T = \frac{1}{2} m \underline{v} \cdot \underline{v} = \frac{1}{2} m[(\Omega R \sin\theta)^2 + (R\dot{\theta}\cos\theta)^2 + (R\dot{\theta}\sin\theta)^2 =$$

$$= \frac{mR^2}{2}\Omega^2 \sin^2\theta + \frac{mR^2}{2}\dot{\theta}^2$$

3. Potential Energy: Taking θ=0 as the datum, we have;

$$V = mgR(1-\cos\theta)$$

4. Lagrangian:

$$L = T - V = \frac{mR^2}{2}\Omega^2 \sin^2\theta + \frac{mR^2}{2}\dot{\theta}^2 - mgR(1-\cos\theta)$$

5. The Variation of Lagrangian:

$$\delta L = \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} = \quad mR^2[\Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta]\delta\theta + mR^2\dot{\theta}\delta\dot{\theta}$$

To apply Hamilton's Principle, we need to express the 2nd term in above equation in terms of δθ. Integrating the 2nd term by parts results:

$$\int_{t_1}^{t_2}\dot{\theta}\delta\dot{\theta}dt = \int_{t_1}^{t_2}\dot{\theta}\frac{d}{dt}(\delta\theta)dt = \left.\cancelto{0}{\dot{\theta}\delta\theta}\,\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\ddot{\theta}\delta\theta dt$$

The integrated term in the above equation vanishes by definition of the variation at the beginning and end of the path. Therefore, applying Hamilton's Principle results:

$$\int_{t_0}^{t_1}\delta L dt = 0 \Rightarrow$$

$$\int_{t_1}^{t_2}[-mR^2\ddot{\theta} + mR^2(\Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta)]\delta\theta dt = 0$$

6. Applying the Hamilton's Principle:

$$\int_{t_0}^{t_1} \delta L dt = 0 \Rightarrow$$

$$\int_{t_1}^{t_2} [-mR^2\ddot{\theta} + mR^2(\Omega^2 \sin\theta \cos\theta - \frac{g}{R}\sin\theta)]\delta\theta dt = 0$$

For the equality to hold, the integrand must vanish at all times. Because δθ is arbitrary, for the integrand to be zero, the coefficient of δθ must be zero. Therefore, the Equation of Motion will results as:

$$\ddot{\theta} + \sin\theta(\frac{g}{R} - \Omega^2 \cos\theta) = 0$$

متشکرم