



INTRODUCTION TO ROBOTICS

(Kinematics, Dynamics, and Design)

SESSION # 13:

MANIPULATOR

INVERSE KINEMATICS

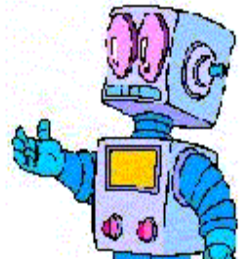
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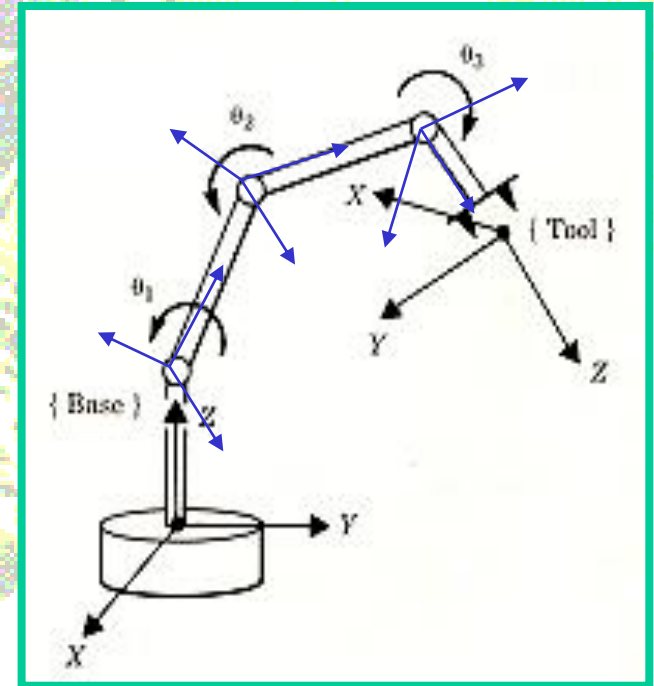
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Inverse Manipulator Kinematics

- **Forward Kinematics:** Describe the position and orientation of the manipulator's end-effector as a function of joint variables relative to a base frame.
- **Inverse Kinematics:** Given the desired position and orientation of the end-effector relative to the base, compute the set of joint variables which will achieve this desired result.



A 3-DOF Manipulator Arm

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

Solving kinematics equations in robotics is a **Non-Linear Problem**.

Given; 0T_n , Find; $\{\theta_1, \theta_2, \dots, \theta_n\}$, is a *non-linear* problem.

Ex: PUMA-560 Robot. Given; 0T_6 , Find; $\{\theta_1, \theta_2, \dots, \theta_6\}$, (see Equation 3.14)

For a **6-DOF** manipulator, we have:

$${}^0T_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **12-Equations, and 6-Unknowns?**
- From **9-Equations** of the Rotation Matrix, only **3-Equations** are independent.
- Therefore, we have **6-independent non-linear equations** and **6-unknowns**.



Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

We have **6-independent non-linear equations** and **6-unknowns**. Therefore, we should investigate the followings:

- **Existence of Solution (وجود جواب).**
- **Multiple Solutions (تعدد جوابها).**
- **Method of Solution (روش حل).**



Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

- **Existence of Solution (وجود جواب) :**

Existence of solution to Inv.-Kin. problem depends on the existence of the specified goal point in the manipulator's **Workspace**.

Workspace/Work-envelope (فضای کاری) : is that volume of space which the end-effector of a robot can reach.

Dexterous Workspace (فضای کاری ماهر) : is that volume of space which the end-effector of a robot can reach with all orientations.

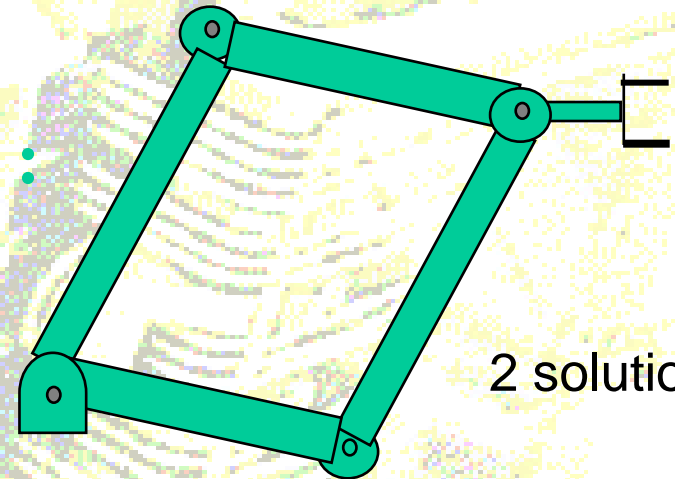


Inverse Manipulator Kinematics

- Solvability (قابل حل بودن):

- ▶ Multiple Solutions (تعدد جوابها) :

A manipulator may reach any position in the interior of its workspace with different configurations. But the system has to be able to choose one.



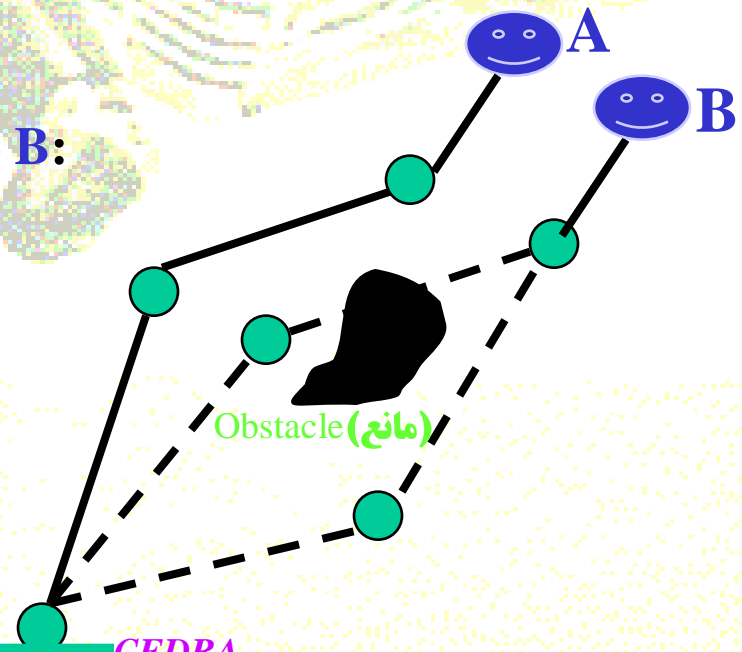
2 solutions!

A manipulator moving from point A to B:

Two solutions exist:

- One causes a **collision**, and
- Other is **safe**.

Therefore, we need to find all solutions.

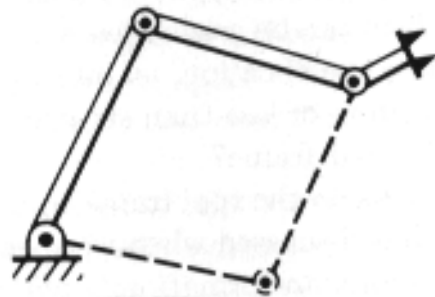


Inverse Manipulator Kinematics

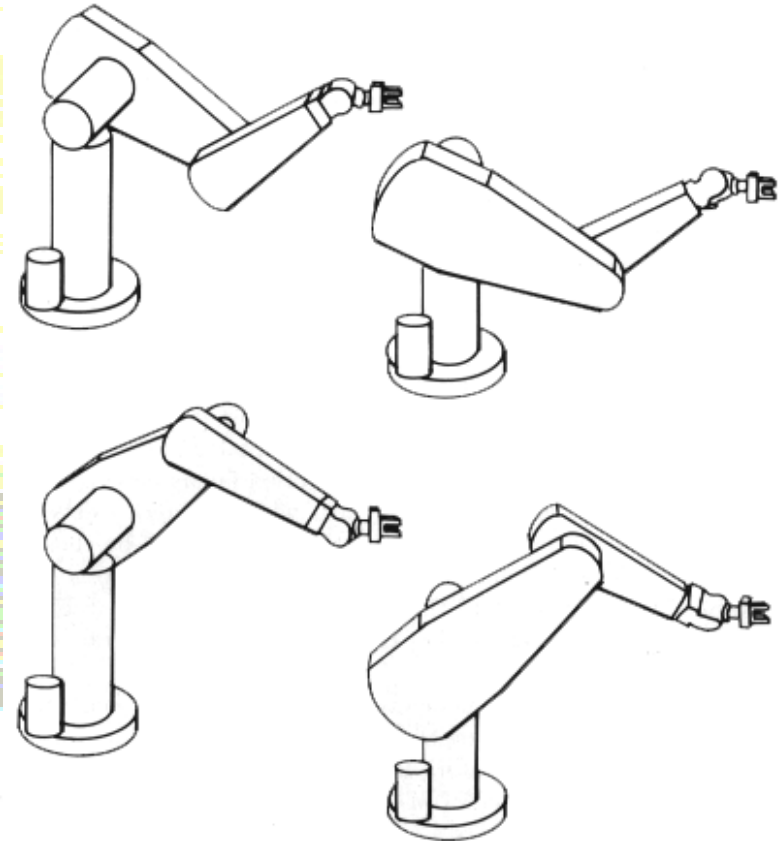
- **Solvability (قابل حل بودن):**

- **Multiple Solutions (تعدد جوابها) :**

Ex: The PUMA-560 manipulator can reach certain goals with **8-different** solutions. Due to the limits imposed on joints ranges, some of these solutions may not be accessible.



Elbow down - Elbow up



{Other 4-solutions are for the wrist}

Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

- **Method of Solution (روش حل):**

Unlike linear equations, no general algorithms exist for solving a set of *non-linear* equations.

A manipulator is considered as **Solvable (قابل حل)**, if it is possible to calculate all its solutions. Two forms of solution strategies exist:

Closed-form-Solutions (حل بسته): Solution method is based on analytical expressions.

Numerical Solutions (حل عددی): Due to their iterative nature, they are too slow, and therefore not a useful approach in solving robot kinematics.



Inverse Manipulator Kinematics

- **Solvability (قابل حل بودن):**

➤ **Method of Solution (روش حل) :**

Since numerical solutions are generally very slow relative to closed form solutions, it is very important to design a manipulator such that a closed form solution exists.

Sufficient condition for a manipulator with **6-Revolute** joints to have a closed-form-solution is that **3-neighboring** joints axes intersect at a point. (**read section 4.6 by Pieper**)

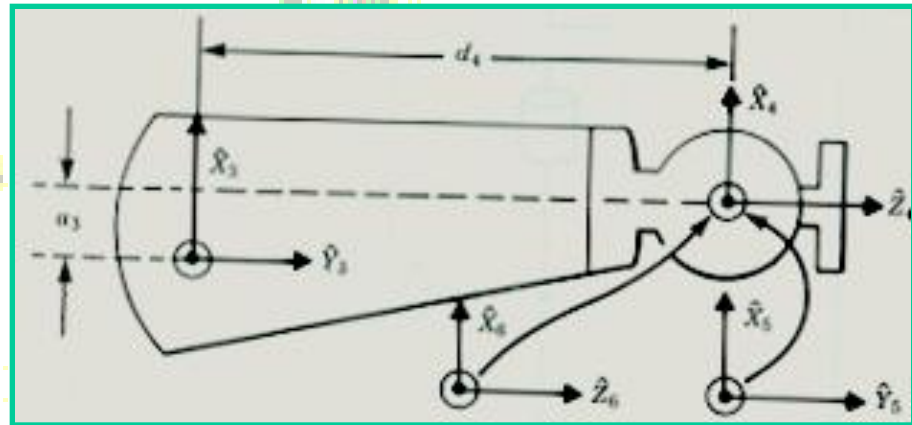
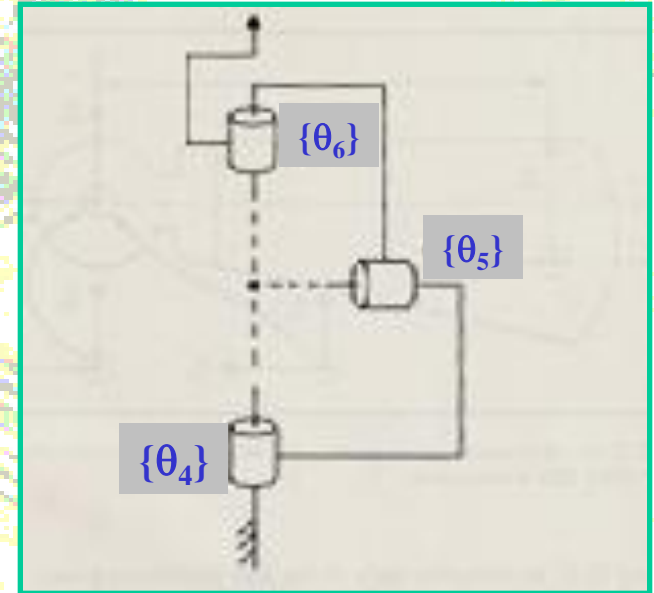
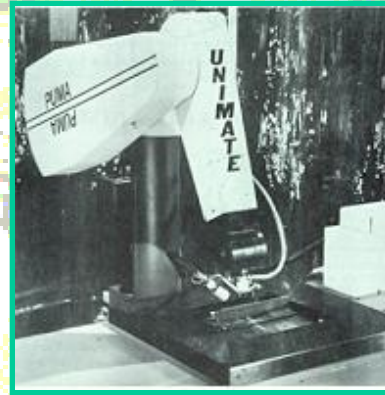
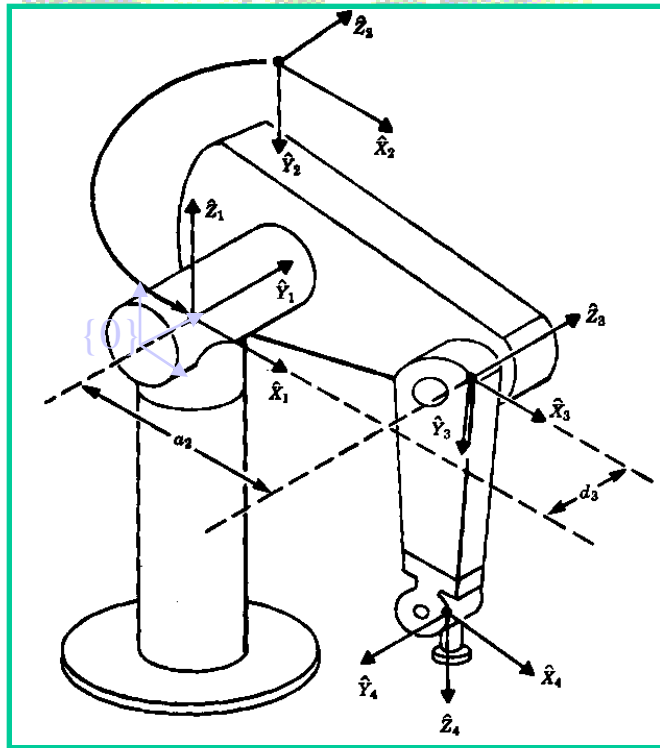
Example: In **PUMA-560**, axes **4**, **5**, and **6** all intersect at a point.



PUMA-560 Manipulator Kinematics

- **Frames Attachment (اتصال چہار چوبہا):**

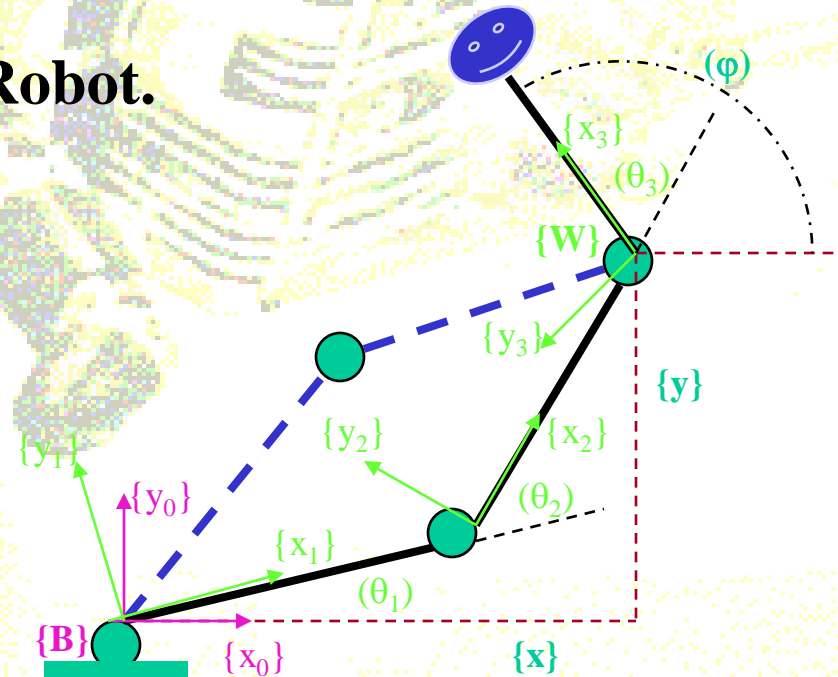
- Joint axes **4**, **5**, and **6** all intersect at a common point.



Inverse Manipulator Kinematics

- **Algebraic Method: No general method exists to solve kinematics equations. Let's solve a few examples.**
- **Ex: A 3-DOF Revolute Planar Robot.**

Joint-i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	$\alpha_0=0$	$a_0=0$	$d_1=0$
2	θ_2	$\alpha_1=0$	$a_1=L_1$	$d_2=0$
3	θ_3	$\alpha_2=0$	$a_2=L_2$	$d_3=0$



Manipulator Kinematics

Example: The 3-link planar manipulator

$${}^0_1T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & l_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & l_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^B_W T = {}^0_3T = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1 C_1 + l_2 C_{12} \\ S_{123} & C_{123} & 0 & l_1 S_1 + l_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

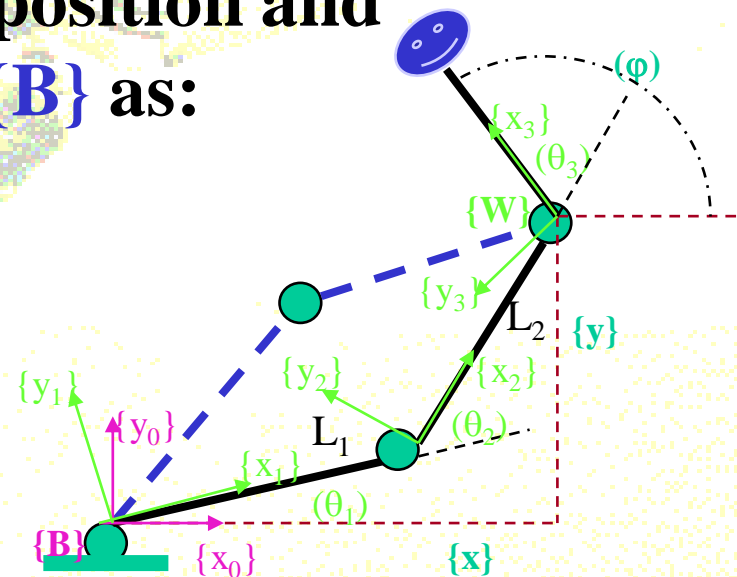
Since this is a planar robot, assume that the **goal point** is a specification of the **{Wrist}** relative to the **{Base}**.



Inverse Manipulator Kinematics

- Therefore, we can use 3-numbers x , y , and φ to specify the goal point such that:
 - x , y : define the origin of frame $\{W\}$, and
 - φ : defines the orientation of $\{W\}$ (3rd-link) relative to the $+x$ axis of the $\{B\}$ frame.
 - Therefore, one can define the position and orientation of $\{W\}$ relative to $\{B\}$ as:

$${}^B T_W = {}^0 T_3 = \begin{bmatrix} C\varphi & -S\varphi & 0 & x \\ S\varphi & C\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(b)}$$



Inverse Manipulator Kinematics

➤ Let us now equate relations (a) and (b) as follows:

$${}^B_W T = {}^0_3 T = \begin{bmatrix} C_{123} & -S_{123} & 0 & \ell_1 C_1 + \ell_2 C_{12} \\ S_{123} & C_{123} & 0 & \ell_1 S_1 + \ell_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\varphi & -S\varphi & 0 & x \\ S\varphi & C\varphi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1) \quad C\varphi = C_{123}$$

$$(2) \quad S\varphi = S_{123}$$

$$(3) \quad x = \ell_1 C_1 + \ell_2 C_{12}$$

$$(4) \quad y = \ell_1 S_1 + \ell_2 S_{12}$$

Square – and – Add – (3) & (4) :

$$x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2\ell_1\ell_2(C_1C_{12} + S_1S_{12})$$

$$\text{Since : } (C_1C_{12} + S_1S_{12}) = C_1(C_1C_2 - S_1S_2) + S_1(S_1C_2 + C_1S_2) = C_2$$

$$x^2 + y^2 = \ell_1^2 + \ell_2^2 + 2\ell_1\ell_2C_2$$



Inverse Manipulator Kinematics

- Using Atan2 function insures finding all solutions.

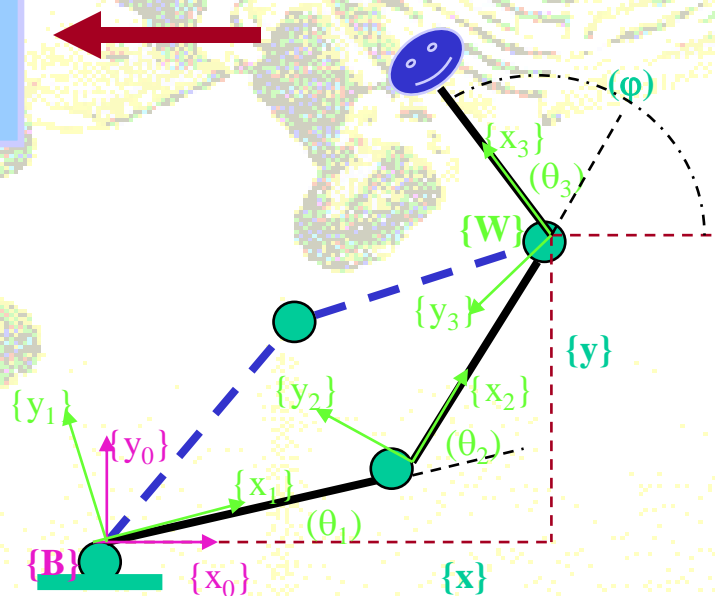
$$C_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

$$S_2 = \pm \sqrt{1 - C_2^2}$$

$$\theta_2 = \text{Atan2}\left(\frac{S_2}{C_2}\right)$$

(This term should be between -1 and 1 . If it is not, that means we have an unreachable point.)

(+ and $-$ means Multiple Solutions for θ_2 :
Elbow-Up and **Elbow-Down** configurations.)



Inverse Manipulator Kinematics

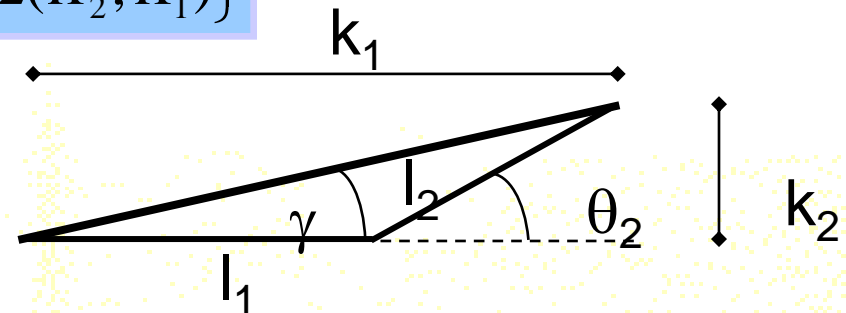
- To find θ_1 use equations (3) and (4) as follows:

$$(3) \quad x = l_1 C_1 + l_2 C_{12} = l_1 C_1 + l_2 C_1 C_2 - l_2 S_1 S_2 = \\ x = (l_1 + l_2 C_2) C_1 - (l_2 S_2) S_1 = K_1 C_1 - K_2 S_1$$

$$(4) \quad y = l_1 S_1 + l_2 S_{12} = K_1 S_1 + K_2 C_1$$

- Let us now change variables to solve these equations:

$$\text{Let : } \begin{cases} K_1 = r \cos \gamma \\ K_2 = r \sin \gamma \end{cases} \Leftrightarrow \begin{cases} r = +\sqrt{K_1^2 + K_2^2} \\ \gamma = \text{Atan2}(K_2, K_1) \end{cases}$$



Now relations for x and y can be expressed as:

Inverse Manipulator Kinematics

➤ Now relations for x and y can be expressed as:

$$x = r \cos \gamma \cos \theta_1 - r \sin \gamma \sin \theta_1 \Rightarrow \frac{x}{r} = \cos(\gamma + \theta_1)$$

$$y = r \cos \gamma \sin \theta_1 + r \sin \gamma \cos \theta_1 \Rightarrow \frac{y}{r} = \sin(\gamma + \theta_1)$$

➤ Therefore:

$$\gamma + \theta_1 = A \tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A \tan 2(y, x) \Rightarrow$$

$$\theta_1 = A \tan 2(y, x) - A \tan 2(K_2, K_1)$$

➤ One solution for θ_1 , and that depends on the sign chosen for θ_2 .

➤ From equations (1) and (2), we can now define θ_3 .

$$\begin{cases} C\varphi = C_{123} \\ S\varphi = S_{123} \end{cases} \Rightarrow \theta_{123} = \theta_1 + \theta_2 + \theta_3 = A \tan 2\left(\frac{S\varphi}{C\varphi}\right) = \varphi \Rightarrow$$

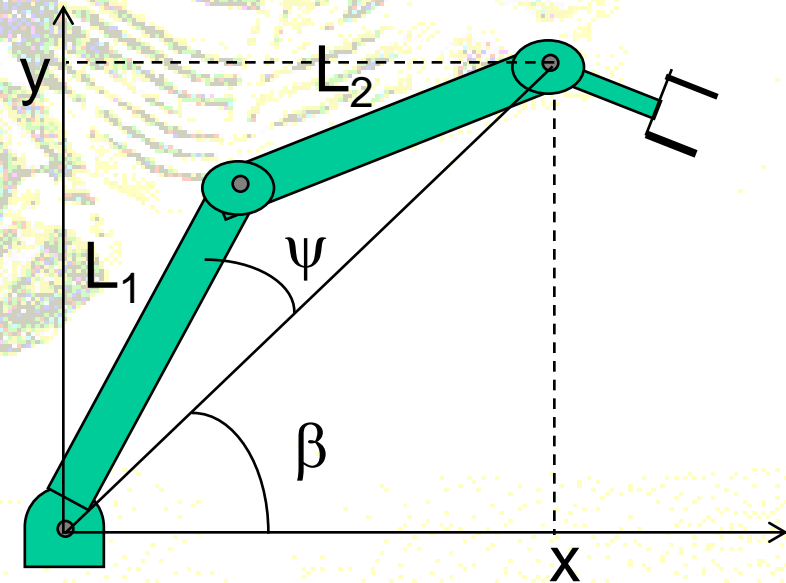
$$\theta_3 = \varphi - \theta_1 - \theta_2$$



Inverse Manipulator Kinematics

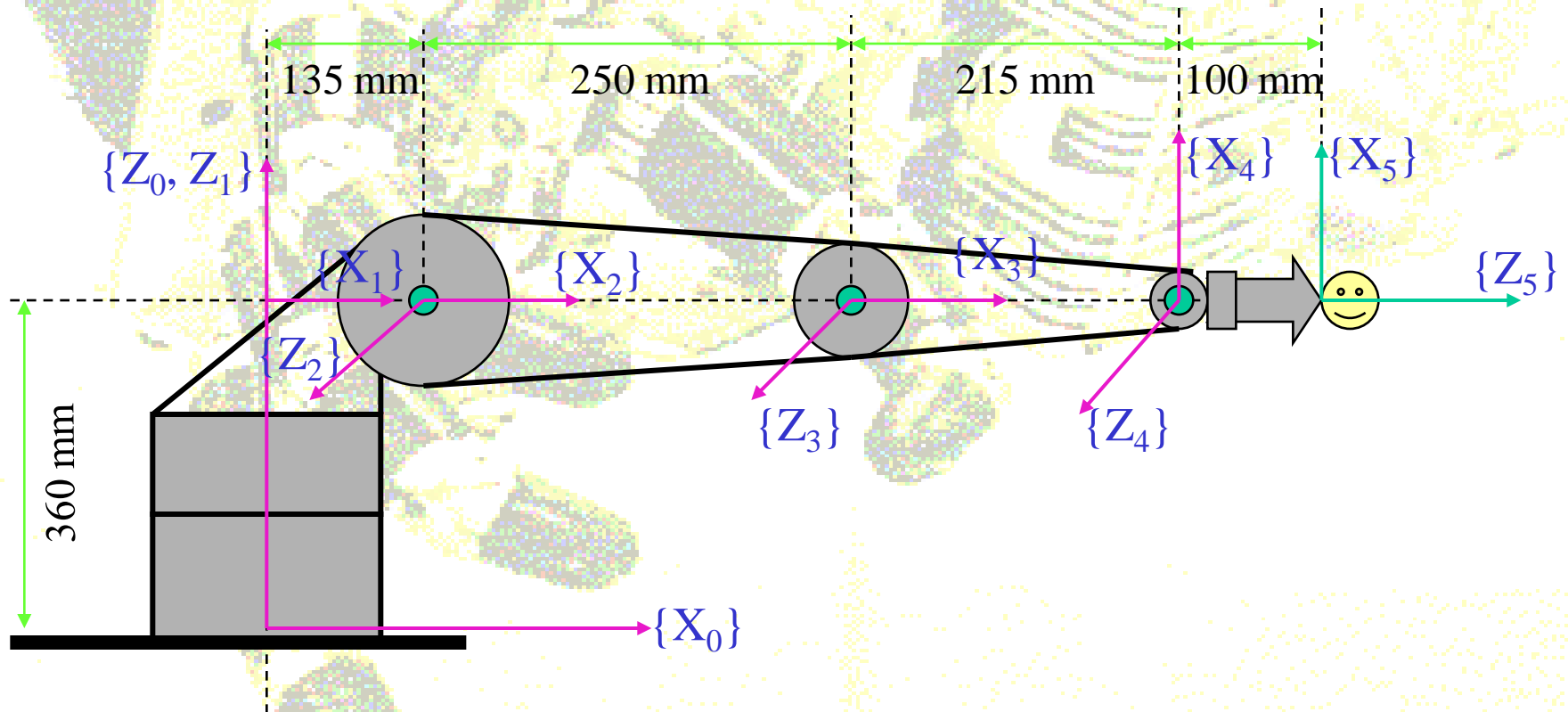
- Geometric Method:** First decompose the spatial geometry of the arm into several plane geometry problems. Then, solve for the joint angles using tools of plane geometry (i.e by applying the “**law of cosines**”). (see book for an example)

Joint-i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	$\alpha_0=0$	$a_0=0$	$d_1=0$
2	θ_2	$\alpha_1=0$	$a_1=L_1$	$d_2=0$
3	θ_3	$\alpha_2=0$	$a_2=L_2$	$d_3=0$



Manipulator Kinematics

- **Example: The Yasukawa/Motoman MK3 Robot.**
 - **A 5-DOF “5R” Revolute Robot**



Manipulator Kinematics

- The Yasukawa/Motoman MK3 Table of Link-Joint Parameters:

Joint-i	${}^{i-1}T_i$	θ_i	α_{i-1}	a_{i-1}	d_i
1	0T_1	θ_1	$\alpha_0 = 0$	$a_0 = 0$	$d_1 = 360$
2	1T_2	θ_2	$\alpha_1 = 90$	$a_1 = 135$	$d_2 = 0$
3	2T_3	θ_3	$\alpha_2 = 0$	$a_2 = 250$	$d_3 = 0$
4	3T_4	θ_4	$\alpha_3 = 0$	$a_3 = 215$	$d_4 = 0$
5	4T_5	θ_5	$\alpha_4 = 90$	$a_4 = 0$	$d_5 = 100$



Yasukawa/Motoman MK3 Manipulator Kinematics

- Now compute each of the link *transformations*:

$${}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} C_3 & -S_3 & 0 & a_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} C_4 & -S_4 & 0 & a_3 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^4_5T = \begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_5 & C_5 & 0 & -d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



YM MK3 Manipulator Kinematics

- Let us now form the 0_5T transformation matrix:

$${}^0_5T = {}^0_1T {}^1_2T {}^2_3T {}^3_4T {}^4_5T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} r_{11} &= C_1 C_{234} C_5 + S_1 S_5 & r_{12} &= -C_1 C_{234} S_5 + S_1 C_5 & r_{13} &= C_1 S_{234} \\ r_{21} &= S_1 C_{234} C_5 - C_1 S_5 & r_{22} &= -S_1 C_{234} S_5 - C_1 C_5 & r_{23} &= S_1 S_{234} \\ r_{31} &= C_5 S_{234} & r_{32} &= -S_5 S_{234} & r_{33} &= -C_{234} \\ p_x &= C_1 (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\ p_y &= S_1 (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\ p_z &= (d_1 + a_2 S_2 + a_3 S_{23} - d_5 C_{234}) \end{aligned}$$



YM MK3 Manipulator

Inverse-Kinematics

- We wish to solve the inverse-kinematics problem yielding $\theta_1 \dots \theta_5$ as a function of $r_{11} \dots r_{33}$, p_x , p_y , p_z .
- Lets start with θ_1 , since no “yaw” motion is present:

$$\theta_1 = A \tan 2\left(\frac{p_y}{p_x}\right) \quad \text{Since} \quad \left\{ \begin{array}{l} p_x = C_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \\ p_y = S_1(a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234}) \end{array} \right\}$$

- Note that we cannot have $A \tan 2(0/0)$!!! If: $p_x = p_y = 0$, then we have a special case.
- We now need $(\theta_2 + \theta_3 + \theta_4)$ to find the wrist center. Note that:

$$r_{13} = C_1 S_{234}, \quad r_{23} = -S_1 S_{234}, \quad r_{33} = -C_{234}$$

- Therefore, we can write:

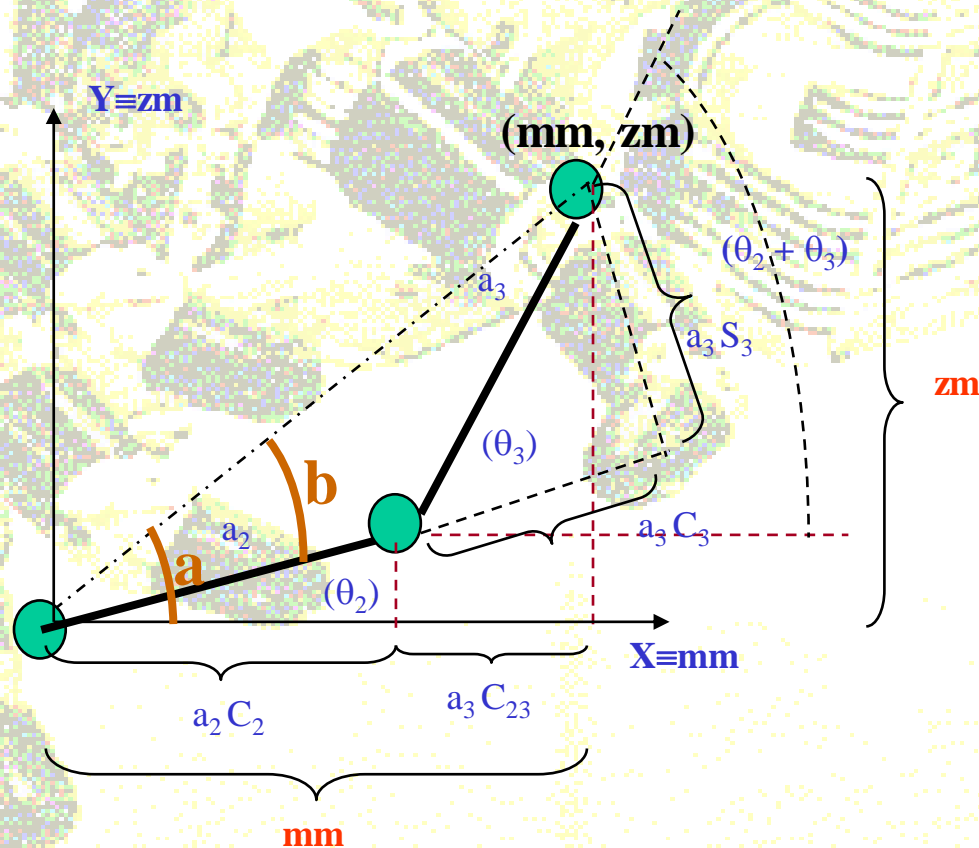
$$C_1 r_{13} - S_1 r_{23} = C_1^2 S_{234} + S_1^2 S_{234} = S_{234} \Rightarrow$$

$$\theta_2 + \theta_3 + \theta_4 = A \tan 2(C_1 r_{13} - S_1 r_{23}, -r_{33})$$



YM MK3 Manipulator Inverse-Kinematics

- Let's now solve for θ_2 and θ_3 as follows (by reconsidering our old planar arm problem):



YM MK3 Manipulator Inverse-Kinematics

Let : $m = C_1 p_x + S_1 p_y = C_1^2(\dots) + S_1^2(\dots) = (a_1 + a_2 C_2 + a_3 C_{23} + d_5 S_{234})$

Let : $X = mm = m - a_1 - d_5 S_{234} = a_2 C_2 + a_3 C_{23}, \quad S_{234} = \text{known}$

Let : $Y = zm = p_z - d_1 + d_5 C_{234} = a_2 S_2 + a_3 S_{23}, \quad C_{234} = \text{known}$

Then :

$$\begin{aligned} mm^2 + zm^2 &= a_2^2 C_2^2 + a_3^2 C_{23}^2 + 2a_2 a_3 C_2 C_{23} + \\ &+ a_2^2 S_2^2 + a_3^2 S_{23}^2 + 2a_2 a_3 S_2 S_{23} = \\ &= a_2^2 + a_3^2 + 2a_2 a_3 (C_2 C_{23} + S_2 S_{23}) = a_2^2 + a_3^2 + 2a_2 a_3 (C_3) \end{aligned}$$

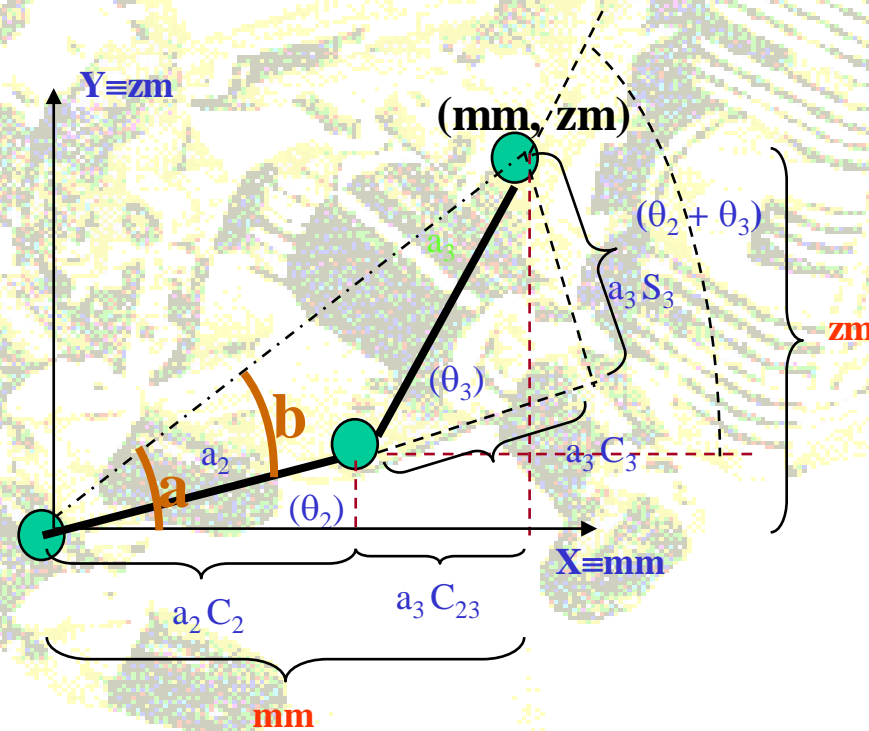
$$\text{Cos } \theta_3 = \frac{mm^2 + zm^2 - a_2^2 - a_3^2}{2a_2 a_3}, \text{ and } \text{Sin } \theta_3 = \pm \sqrt{1 - C_3^2}$$

$$\theta_3 = \text{A tan } 2\left(\frac{S_3}{C_3}\right)$$



YM MK3 Manipulator Inverse-Kinematics

★ Considering the following figure again, we have:



$$\left\{ \begin{array}{l} a = A \tan 2(zm, mm) \\ b = A \tan 2(a_3 S_3, a_2 + a_3 C_3) \end{array} \right\} \Rightarrow \theta_2 = a - b \quad \leftarrow$$



YM MK3 Manipulator Inverse-Kinematics

- Solving for θ_4 we have:

$$\theta_4 = (\theta_2 + \theta_3 + \theta_4) - \theta_2 - \theta_3$$

- Finally, to solve for θ_5 , given $\theta_1 \dots \theta_4$, note that 0_4T is now known. Therefore, we can write the following equation:

$${}^0_5T = {}^0_4T {}^4_5T \Rightarrow {}^4_5T = {}^0_4T^{-1} {}^0_5T \Rightarrow$$

$$\begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ 0 & 0 & -1 & -d_5 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 C_{234} & S_1 C_{234} & S_{234} & \dots \\ -C_1 S_{234} & -S_1 S_{234} & C_{234} & \dots \\ S_1 & -C_1 & 0 & \dots \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence: } \begin{cases} S_5 = S_1 r_{11} - C_1 r_{21} \\ C_5 = S_1 r_{12} - C_1 r_{22} \end{cases} \Rightarrow \theta_5 = A \tan 2(S_5, C_5) \quad \leftarrow$$



Chapter 4 Exercises:

- 4.1, 4.2, 4.3, 4.8, 4.9
- 4.1 Programming Exercise
- 4.1 MathLab Exercise
- Programming of the PUMA 560
Inverse Kinematics

