Rent Division Among Groups

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Abstract. In this paper, we extend the *Rent Sharing* problem to the case that every room must be allocated to a group of agents. In the classic Rent Sharing problem, there are n agents and a house with n rooms. The goal is to allocate one room to each agent and assign a rent to each room in a way that no agent envies any other option. Our setting deviates from the classic Rent Sharing problem in a sense that the rent charged to each room must be divided among the members of the resident group.

We define three notions to evaluate fairness, namely, *weak envy-freeness*, *aggregate envy-freeness* and *strong envy-freeness*. We also define three different policies to divide the cost among the group members, namely, *equal, proportional,* and *free* cost-sharing policies.

We present several positive and negative results for different combinations of the fairness criteria and rent-division policies. Specifically, when the groups are pre-determined, we propose a *strong envy-free* solution that allocates the rooms to the agents, with free cost-sharing policy. In addition, for the case that the groups are not pre-determined, we propose a strong envy-free allocation algorithm with equal cost-sharing policy. We leverage our results to obtain an algorithm that determines the maximum total rent along with the proper allocation and rent-division method.

Keywords: Fairness; Envy-freeness; Rent Sharing; House Allocation

1 Introduction

Envy-freeness is a famous notion and a central concept studied extensively since 1960's in the literature of economics [1–8]. An allocation is envy-free, if every agent prefers his allocated share to that of other players.

Envy-free resource allocation is studied for various types of resources. In the setting we study, the resources are a set of indivisible goods (rooms) along with one divisible bad (money). Although there are many different real-life applications that fit into this setting, here, we use the terminology of Rent Sharing. In the well known *Rent Sharing* problem, n agents are willing to rent a house with n rooms, and one seeks to somehow allocate the rooms to the agents and determine the rent of the rooms so that each agent prefers his own option. The challenge in this problem is that the rooms are heterogeneous and the agents have different valuations over the rooms. Thus, to maintain fairness, the rent must be wisely divided.

Formally, let $\mathcal{H}$ be a house with n rooms and let $v_{i,j}$ be the value of room j for agent a_i. The utility of a_i for renting room j at price r_j is $u_{i,j} = v_{i,j} - r_j$. Agent a_i (weakly) prefers room j to k, if $u_{i,j} \geq u_{i,k}$. In the *Rent Sharing* problem, we would like to charge a rent to each room and allocate one room to each agent, such that the resulting allocation preserves envy-freeness, i.e., every agent prefers his option. The most interesting aspect of the Rent Sharing problem is that with mild assumptions, an envy-free solution is always guaranteed to exist.

The existence of an envy-free solution for the Rent Sharing problem is proved by Su [8]. In addition, Aragones [9] proposed a polynomial time algorithm to find an envy-free solution. The solution is not necessarily unique, and there may be several envy-free allocations. Therewith, we can optimize other objectives among the feasible solutions. For example, Gal, Mash, Procaccia, and Zick [6] consider the problem of finding an envy-free solution that maximizes the value of agent with the minimum utility to his room (maximin solution).

The basic assumption in the classic Rent Sharing problem is that every room must be allocated to a single agent. However, there are situations that this assumption is no longer applicable. For example in dormitories, each room is allocated to a group of students. As another example (albeit, not in the terminology of Rent Sharing) think of the following scenario: a set of n tasks that must be performed by a set of workers having different skills and interests. In order to perform each task, we must assign it to a group of workers. How can we fairly assign the tasks to the groups and pay the agents? One can think of tasks as rooms and workers as agents and their salary as their rent share.

In such situations, a new challenge arises: the agents within a group may have diverse valuations, i.e., a room might be acceptable to some of the agents in a group but not by the others. In this paper, our goal is to discuss such situations.

Many previous studies consider the fair division problem among groups of agents [10–17]. The groups can be either pre-determined or be formed by the algorithm. The majority of the article focuses on the case that the groups are pre-determined. In this model, there is a set $\mathcal{G}$ of groups, with each group g_i consisting of m_i agents and each agent having a value function over the rooms. Our goal is to fairly allocate a room to each group and divide the rent among the agents. We name this problem *Group Rent Sharing*.

In Section 2.1, we introduce three notions for evaluating fairness: *aggregate envy-freeness*, *weak envy-freeness*, and *strong envy-freeness*. Recall that every solution for the Group Rent Sharing problem must determine a method by which the rent is divided among the residents of a room. We name this dividing method *cost-sharing policy*. In Section 2.2, different possible policies and their relations to various fairness criteria are discussed. We define three policies: *equal, proportional and free* and study the consistency of these policies and fairness notions.

After all we consider the case where the groups are not pre-determined. In this case, the allocation algorithm must also partition the agents into groups. We propose a strong envy-free allocation with an equal cost-sharing policy in Section 4.

As in the classic rent sharing problem, along with fairness, we can consider optimizing other objectives. Here, we seek to maximize the total rent. Note that the allocation must be individually rational, i.e., the utility of every agent in the final solution must be non-negative. The summary of our results for different fairness criteria and cost-sharing policies can be found in Table 2.

1.1 Related Works

Previous works that are related to ours fall into two categories: a stream of studies on the Rent Sharing problem and the works that consider fair division among groups.

Fair division of resources is widely studied in the context of economics and mathematics where the problem mostly considers cases with either single divisible item (also known as cake-cutting)[18, 19, 5, 4, 7, 20, 21], or a set of indivisible items [22–28]. In addition, a combination of these two settings is studied, where there is a set of indivisible items together with a single divisible resource. The rent division problem is, in fact, a combination of a divisible resource (money) and a set of indivisible items (rooms).

The problem of fairly dividing indivisible items with money was firstly introduced by Alkan, Demange, and Gale [22]. They show that for a sufficiently large amount of money, an envy-free allocation exists. They also suggest optimizing other objectives over myriad possible envy-free solutions. Specifically, they introduce the *money Rawlsian* solution in which the goal is to maximize the minimum money taken from every agent. Aragones [9], suggests an algorithm for computing a money Rawlsian envy-free solution in polynomial time. He also shows that every envy-free solution preserves envy-freeness if we re-allocate the rooms by the welfare-maximizing allocation.

Su [8] explains the *Sperner's Lemma* and describes its applications to fair division problems. Especially, he investigates the Rent Sharing problem and used Sperner's Lemma to show the existence of an envy-free allocation. Procaccia, Velez, and Yu [29] extend the classic rent sharing problem by considering a budget for each agent. They study the conditions under which an envy-free allocation with given budget constraints is possible and propose an algorithm to find one. Gal, Mash, Procaccia, and Zick [6] have recently conducted a study on finding equitable and maximin envy-free allocations. The former is the envy-free allocation that minimizes the disparity (the maximum difference) of the agents' utilities and the latter aims to maximize the minimum utility of the agents. They show that a maximin allocation is also equitable. Then, they propose an LP-based method to compute these allocation in polynomial time.

Some works consider the fair allocation problem among groups or families, for example Segal-Halevi and Nitzan [11] consider the proportional allocation for the case that the resource must be divided among families. They introduce three notions to evaluate fairness for this case, namely, *Average, Unanimous* and *Democratic* proportionality and show various results for these notions. Chan et.al. [10] consider the Rent Sharing problem for the case where every room must be allocated to 2 agents. In their model, the groups are not known in advance. They define various solution concepts and study the complexity of their corresponding search problem. In contrast to our work, they do not consider any cost-sharing policy.

From a practical point of view, there are a considerable number of empirical studies that consider notion of fairness between groups rather than individuals [12–15, 30], mostly in the context of ultimatum games.

2 Model Definition and Preliminaries

We refer to the rooms by their indices and denote the set of Groups by $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$. Furthermore we suppose that each group g_i consists of m_i agents and denote the j'th agent of g_i by $a_{i,j}$. The value of room k for agent $a_{i,j}$ is denoted by $v_{i,j,k}$. In this paper we suppose that the valuations are normalized, so that for each agent $a_{i,j}$, $\sum_k v_{i,j,k} = \frac{1}{m_i}$. Hence, the total value of each group for the house is 1. In the *Group Rent Sharing* problem, we seek to find a triple $\mathcal{S} = (\mathcal{A}, \mathcal{R}, \mathcal{D})$ where:

- $\mathcal{A} : \mathcal{G} \to [n]$ [III] is a bijection that allocates one room to each group.
- $\mathcal{R} : [n] \to \mathbb{R}_{\geq 0}$ is a rent division function that determines the rent of each room.
- $\mathcal{D} : (\mathcal{N}, [n]) \to \mathbb{R}$ [IV], is a cost-sharing function where $\mathcal{D}(a_{i,j}, k)$ determines the cost assigned to agent $a_{i,j}$ for living in room k. Each cost-sharing function must have the property that the total amount of cost assigned to each group a room must be exactly equal to the rent determined for that room, i.e., for all i, $\sum_{j=1}^{m_i} \mathcal{D}(a_{i,j}, k) = \mathcal{R}(k)$ for every k. We use $d_{i,j,k}$ to refer to the value of $\mathcal{D}(a_{i,j}, k)$.

We refer to such a triple as allocation-triple. Roughly, in every allocation-triple, one should determine the room that must be allocated to each group ($\mathcal{A}$), the rent charged to each room ($\mathcal{R}$) and the way that rent is divided among the agents in each group for each room ($\mathcal{D}$).

When the groups are not pre-determined, we suppose that $\mathcal{N}$ is the set of agents and the allocation algorithm must determine a quadruple $\mathcal{S} = (\mathcal{B}, \mathcal{A}, \mathcal{R}, \mathcal{D})$, where $\mathcal{B}$ is a function that allocates a group to each agent. The allocation function $\mathcal{B}$ has the restriction that each group must contain exactly m agents. Functions $\mathcal{A}$, $\mathcal{R}$, and $\mathcal{D}$ are defined similar to the Group Rent Sharing problem.

2.1 Fairness Criteria

Our goal in this section is to extend the notion of envy-freeness to the case that every room must be allocated to a group of agents. For this, we define three notions to evaluate fairness: aggregate envy-freeness, weak envy-freeness, and strong envy-freeness. Fix an allocation-triple $\mathcal{S} = (\mathcal{A}, \mathcal{R}, \mathcal{D})$ (or quadraple $\mathcal{S} = (\mathcal{B}, \mathcal{A}, \mathcal{R}, \mathcal{D})$ when the groups are not pre-determined). Denote by $u_{i,j,k}$, the utility of agent $a_{i,j}$, if room k is allocated to group g_i, regarding cost-sharing function $\mathcal{D}$, i.e., $u_{i,j,k} = v_{i,j,k} - d_{i,j,k}$. In this paper, we consider the allocation triples which are *individual rational*, meaning that every agent recieves a non-negative utility.

Definition 1. *$\mathcal{S}$ is* weak envy-free*, if for every k, there exists at least one agent in room k that does not envy any other option.*

In other words, $\mathcal{S}$ is weak envy-free if for each group g_i, at least one agent $a_{i,j}$ in g_i does not envy any other room, which means for all $k \neq \mathcal{A}(g_i)$ we have $u_{i,j,\mathcal{A}(g_i)} \geq u_{i,j,k}$.

[III] $[n]$ refers to the set $\{1, 2, \ldots, n\}$.
[IV] $\mathcal{N} = \bigcup_{i,j} a_{i,j}$

One shortcoming of this notion is that it does not consider the preferences of everyone and hence, an allocation may be unfair to all the agents in a group except one. Our next notion seeks to somehow resolve this issue.

Definition 2. *$\mathcal{S}$ is* aggregate envy-free*, if the total utility of agents in each room is at least as large as their total utility for any other room, i.e., for every group g_i,*

$$\sum_j u_{i,j,\mathcal{A}(g_i)} \geq \sum_j u_{i,j,k} \qquad \forall k.$$

To put it simply, if we consider each group g_i as an agent a_i with value $\sum_j v_{i,j,k}$ for each room k, we want to find an envy-free solution for the classic Rent Sharing problem with agents $\{a_1, a_2, \ldots, a_n\}$ and rooms in $\mathcal{H}$. In fact, this notion takes the aggregate utility of a group into account, instead of considering only one agent from each group. However, the aggregate utility of a group does not capture the utility of every individual, i.e., some of the agent may still be unsatisfied. In the strong envy-free notion, we desire to satisfy all the agents.

Definition 3. *$\mathcal{S}$ is* strong envy-free*, if for each agent $a_{i,j}$ and every $k \neq \mathcal{A}(g_i)$,*

$$u_{i,j,\mathcal{A}(g_i)} \geq u_{i,j,k}.$$

2.2 Cost-sharing

Recall that in the classic *Rent Sharing* problem, we desire to charge a rent to each room. For the Group Rent Sharing problem, in addition to a method for dividing the rent, we need to formulate a policy to split the rent among the agents in each room. As mentioned before, we denote such a policy by $\mathcal{D}$. In this section, we discuss on different possible polices for dividing the rent of a room among the resident agents.

The first and the easiest solution that immediately bears in mind, is to split the rent equally among the agents in each group. We name such a cost-sharing policy, *equal cost-sharing*.

Definition 4. *A cost-sharing policy $\mathcal{D}$ is equal, if the cost assigned to each agent for a room equals to his roommates, i.e., for each group g_i and each room k and agents $a_j, a_{j'}$ we have $d_{i,j,k} = d_{i,j',k}$.*

Even though this policy seems natural, considering the fact that the agents in one group may have diverse interests in a room, rent discrimination would be more reasonable. One idea is that each agent pays a price proportional to his valuation for that room.

Definition 5. *A cost-sharing policy $\mathcal{D}$ is proportional, if for each group g_i, agent $a_{i,j}$ pays the rent $d_{i,j,k} = \frac{v_{i,j,k}}{V_{i,k}}\mathcal{R}(k)$ for room k, where $V_{i,k} = \sum_{j'} v_{i,j',k}$.* [V]

[V] Note that if $V_{i,k} = 0$, by individual rationality, $\mathcal{R}(k) = 0$ and no agent has to pay any cost.

As we discuss in Section 3, the proportional cost-sharing policy is inconsistent to strong envy-freeness, i.e., there are cases that no strong envy-free allocation exists with respect to proportional cost-sharing policy.

The final policy we introduce considers no restriction on the rent charged to each agent for each room. We call such a method *free cost-sharing* policy. The only criterion for a free cost-sharing policy is that the total payment of the agents in each group for a specific room must sum up to the rent fixed for that room. The main results of this paper are concerned with the strong envy-freeness and free cost-sharing policy. Prior to explaining our main results, we will justify the model and subsequently introduce some possible and impossible results in pertaining to our model.

3 Pre-determined Groups

As elaborated earlier, our model comprises two essential ingredients: the fairness criterion and the cost-sharing policy. In this section, we shed light on the relation between these two components. To do so, we define the concept of consistency.

Definition 6. *A fairness criterion $\mathcal{F}$ is consistent with cost-sharing function $\mathcal{D}$, if for every instance of the* Group Rent Sharing *problem, an allocation-triple $\mathcal{S}$ with cost-sharing function $\mathcal{D}$ exists, such that $\mathcal{S}$ preserves $\mathcal{F}$.*

In Lemmas 1 and 2, we show, through counter-examples, that some fairness criteria and some cost-sharing policies are inconsistent.

Lemma 1. *For the case that the groups are pre-determined, the equal cost-sharing policy and strong envy-freeness are inconsistent.*

Proof. Consider the following instance: let $|\mathcal{G}| = 2n$, $g_i = \{a_{i,1}, a_{i,2}\}$ and there are $2n$ rooms. Furthermore, suppose that the valuations of the agents in each group g_i are as follows:

$$v_{i,1,k} = \begin{cases} \frac{1}{3n} & k \leq n \\ \frac{1}{6n} & k > n \end{cases} \qquad v_{i,2,k} = \begin{cases} \frac{1}{6n} & k \leq n \\ \frac{1}{3n} & k > n \end{cases}$$

Consider an arbitrary rent division function $\mathcal{R}$. Due to the symmetric construction of the valuations, we can observe w.l.o.g. that the room 1 is the one with the maximum rent and is assigned to g_1. Let $p = \mathcal{R}(1)$ be the maximum rent. Considering the fact that the cost-sharing policy is equal, the utility of the agents in g_1 would be $u_{1,1,1} = \frac{1}{3n} - \frac{p}{2}$ and $u_{1,2,1} = \frac{1}{6n} - \frac{p}{2}$. Note that the rent assigned to room $i \leq n$ must exactly equal p. Otherwise, agent $a_{1,1}$ envies that room. Now, suppose that $\mathcal{R}(2n) = q$. Thus, the utility of the agents in group g_1, when room $2n$ is allocated to g_1 would be $u_{1,1,2n} = \frac{1}{6n} - \frac{q}{2}$ and $u_{1,2,2n} = \frac{1}{3n} - \frac{q}{2}$. Since we intend our allocation to be strong envy-free, we have:

$$u_{1,1,1} \geq u_{1,1,2n} \Rightarrow \frac{1}{3n} - \frac{p}{2} \geq \frac{1}{6n} - \frac{q}{2} \tag{1}$$

$$u_{1,2,1} \geq u_{1,2,2n} \Rightarrow \frac{1}{6n} - \frac{p}{2} \geq \frac{1}{3n} - \frac{q}{2} \tag{2}$$

$$(1), (2) \Rightarrow q \geq p \tag{3}$$

As p is the maximum rent:

$$\Rightarrow q = p \tag{4}$$

$$(2),(4) \Rightarrow \frac{1}{n} \leq 0$$

Which contradicts $n > 0$.

The counter-example described in the proof of Lemma 1 is independent of $\mathcal{R}$ which means even for a very large amount of rent, a strong envy-free allocation-triple with the equal cost-sharing policy is impossible. In Lemma 2, we show that the proportional cost-sharing policy and strong envy-freeness are also inconsistent.

Lemma 2. *For the case that the groups are pre-determied, the proportional cost-sharing policy and strong envy-freeness are inconsistent.*

Due to lack of space, we omit the proof of Lemma 2[VI] but, to give an intuition, take into account the following instance: let $|\mathcal{G}| = 3$ and let $g_i = \{a_{i,1}, a_{i,2}\}$. Moreover, suppose that the valuation functions of the agents are as in Table 1. As we illustrate in the proof of Lemma 2, for this instance, no allocation of rooms can guarantee strong envy-freeness with the proportional cost-sharing policy. To show this, we consider different allocation possibilities and show that in each of them, at least one agent envies another choice.

Table 1: Valuation of the agents

	1	2	3
$a_{1,1}$	0	3/8	1/8
$a_{1,2}$	3/8	0	1/8
$a_{2,1}$	0	0	1/2
$a_{2,2}$	1/2	0	0
$a_{3,1}$	0	1/4	1/4
$a_{3,2}$	0	1/4	1/4

Observation 1 *Aggregate envy-freeness is implied by strong envy-freeness . Furthermore, aggregate envy-freeness implies weak envy-freeness.*

In Lemmas 3 and 4, we show that both equal and proportional cost-sharing policies are consistent to aggregate envy-freeness. Moreover, Considering Observation 1, both the policies are consistent to weak envy-freeness as well.

Lemma 3. *The equal cost-sharing policy and aggregate envy-freeness are consistent.*

Lemma 4. *The proportional cost-sharing policy and aggregate envy-freeness are consistent.*

The general idea behind proving both Lemmas 3 and 4 is to build a classic Rent Sharing problem by aggregating the valuations of the agents in each group and then showing that dividing the rent by each of these two policies preserves aggregate envy-freeness for every group.

[VI] We refer the reader to the full version of the paper for this proof.

3.1 Strong Envy-freeness and Free Cost-sharing

This section deals with the results surrounding the strong envy-free allocations with the free cost-sharing policy. As described in Section 3, proportional and equal cost-sharing policies are not consistent with strong envy-freeness. Here, we show that with the free cost-sharing policy, one can find a strong envy-free allocation (Lemma 3). Our assumption in this section is that the groups are known in advance. However, the results can be trivially extended to the case that the groups are not pre-determined.

We start this section with Observation 2, which indicates that increasing the rent for all the agents preserves envy-freeness. We use Observation 2 as a basis upon which the proof of Theorem 3 is obtained.

Observation 2 *Let $\mathcal{S} = (\mathcal{A}, \mathcal{R}, \mathcal{D})$ be a strong envy-free allocation-triple and let c be a constant. Furthermore, let $\mathcal{S}^* = (\mathcal{A}, \mathcal{R}^*, \mathcal{D}^*)$ be an allocation-triple such that for all k, $\mathcal{R}^*(k) = \mathcal{R}(k) + c$ and for every agent $a_{i,j}$, $d^*_{i,j,k} = d_{i,j,k} + \frac{c}{m_i}$. Then, $\mathcal{S}^*$ is also strong envy-free.*

Theorem 3. *Strong envy-freeness is consistent with the free cost-sharing policy.*

Proof. Consider a proxy agent a_i for each group g_i and set the valuation of a_i for room k as $v_{i,k} = \sum_{j=1}^{m_i} v_{i,j,k}$. Now, consider the classic *Rent Sharing* problem instance with agents $a_1, a_2, \ldots, a_n$ and house $\mathcal{H}$. We know that the utility of a_i for room k is $u_{i,k} = v_{i,k} - \mathcal{R}(k)$. On the other hand, we know that an envy-free allocation for this instance always exists [8]. Thus, we can find an allocation $\mathcal{A}^*$ and a rent division function $\mathcal{R}^*$ such that for each proxy agent a_i and any $k \neq \mathcal{A}^*(g_i)$, $u_{i,\mathcal{A}^*(g_i)} \geq u_{i,k}$, which means $v_{i,\mathcal{A}^*(g_i)} - \mathcal{R}^*(\mathcal{A}^*(g_i)) \geq v_{i,k} - \mathcal{R}^*(k)$. By definition, for all $k \neq \mathcal{A}^*(g_i)$,

$$\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \mathcal{R}^*(\mathcal{A}^*(g_i)) \geq \sum_{j=1}^{m_i} v_{i,j,k} - \mathcal{R}^*(k). \tag{5}$$

Now, consider the allocation-triple $\mathcal{S}^* = (\mathcal{A}^*, \mathcal{R}^*, \mathcal{D}^*)$, where the cost-sharing function $\mathcal{D}^*$ is determined as follows:

$$d^*_{i,j,k} = v_{i,j,k} - \frac{\sum_{t=1}^{m_i} v_{i,t,k} - \mathcal{R}^*(k)}{m_i} \tag{6}$$

Note that $\sum_j d^*_{i,j,k} = \mathcal{R}^*(k)$. We claim that $\mathcal{S}^*$ is strong envy-free. To show this, take an arbitrary agent $a_{i,j}$. We have $u_{i,j,k} = v_{i,j,k} - d^*_{i,j,k}$, which means

$$u_{i,j,k} = v_{i,j,k} - (v_{i,j,k} - \frac{\sum_{t=1}^{m_i} v_{i,t,k} - \mathcal{R}^*(k)}{m_i}).$$

Notice that $u_{i,j,\mathcal{A}^*(g_i)} = v_{i,j,\mathcal{A}^*(g_i)} - d^*_{i,j,\mathcal{A}^*(g_i)}$. Regarding Equation (6),

$$\begin{aligned}
u_{i,j,\mathcal{A}^*(g_i)} &= v_{i,j,\mathcal{A}^*(g_i)} - \big(v_{i,j,\mathcal{A}^*(g_i)} \\
&\quad - \frac{\sum_{t=1}^{m_i} v_{i,t,\mathcal{A}^*(g_i)} - \mathcal{R}^*(\mathcal{A}^*(g_i))}{m_i}\big) \\
&= \frac{\sum_{t=1}^{m_i} v_{i,t,\mathcal{A}^*(g_i)} - \mathcal{R}^*(\mathcal{A}^*(g_i))}{m_i} \\
&\geq \frac{\sum_{t=1}^{m_i} v_{i,t,k} - \mathcal{R}^*(k)}{m_i} \\
&\geq v_{i,j,k} - (v_{i,j,k} - \frac{\sum_{t=1}^{m_i} v_{i,t,k} - \mathcal{R}^*(k)}{m_i}) \\
&\geq v_{i,j,k} - d^*_{i,j,k} = u_{i,j,k}
\end{aligned}$$

Thus far, we've shown that $\mathcal{S}^*$ is strong envy-free. However, in $\mathcal{S}^*$, there may be agents with negative utilities. Let $u_{min} = \min_{i,j} v_{i,j,\mathcal{A}^*(g_i)} - d^*_{i,j,\mathcal{A}^*(g_i)}$ and let $Z = \max_i |g_i|$. Note that if $u_{min} > 0$, then the individual rationality constraint has already fulfilled. Otherwise, let $\mathcal{R}^{**}$ be the rent function such that for every room k, $r^{**}(k) = r^*(k) + u_{min} \cdot Z$ and let $\mathcal{D}^{**}$ be a function such that for all i, j, k, $d^{**}_{i,j,k} = d^*_{i,j,k} + \frac{Z \cdot u_{min}}{|g_i|}$. By Observation 2, $\mathcal{D}^{**}$ is also strong envy-free with nonnegative utilities and hence guarantees individual rationality. In summary, value of $d^{**}_{i,j,k}$ would be

$$v_{i,j,k} - \frac{\sum_{t=1}^{m_i} v_{i,t,k} - \mathcal{R}^*(k)}{m_i} + \frac{Z \cdot (\min_{w,t} v_{w,t,\mathcal{A}^*(g_w)} - d^*_{w,t,\mathcal{A}^*(g_w)})}{|g_i|}.$$

In light of Theorem 3, we can present an algorithm for computing a strong envy-free allocation-triple. We already know that a solution to the classic rent sharing problem can be found in polynomial time. All the other steps described in Theorem 3 can be easily implemented in polynomial time. Thus, a strong envy-free solution with the free cost-sharing policy can be found in polynomial time.

The idea to find a solution with maximum possible total rent is inspired by [6]. Let $\mathcal{A}^*$ be an allocation, which is welfare-maximizing. In [6], it is shown that if an envy-free solution exists with arbitrary allocation function $\mathcal{A}$ and rent sharing function $\mathcal{R}$, then the pair $\mathcal{A}^*$ and $\mathcal{R}$ is also envy-free. In Theorem 4, we use a generalized form of this statement to obtain a strong envy-free allocation. In fact, we show that if $\mathcal{S} = (\mathcal{A}, \mathcal{R}, \mathcal{D})$ is a strong envy-free allocation-triple, then so is $\mathcal{S}' = (\mathcal{A}^*, \mathcal{R}, \mathcal{D})$.

Theorem 4. *A strong envy-free allocation-triple with the free cost-sharing policy that maximizes total rent can be found in polynomial time.*

Proof. Recall the definition of proxy agent from the proof of Theorem 3 . An allocation $\mathcal{A}$ is welfare-maximizing, if it maximizes value of the following expression:

$$\sum_{i=1}^{|\mathcal{G}|} \sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} = \sum_{i=1}^{|\mathcal{G}|} v_{i,\mathcal{A}(a_i)}.$$

Such an allocation can be found in polynomial time by finding a maximum weighted matching in the bipartite graph representing the tendency of the proxy agents to the rooms, i.e. the weight of the edge between proxy agent i and room k is $\sum_{j=1}^{m_i} v_{i,j,k}$. Now, consider the pseudo-code described in Algorithm 1. The algorithm begins with computing a welfare-maximizing allocation of the rooms to the proxy agents. Let $\mathcal{A}$ be the welfare-maximizing allocation. we find the desired allocation-triple by solving a linear program described in Algorithm 1, which computes the envy-free allocation with the maximum possible price. In this LP, the first set of constraints ensures that the sum of the costs assigned to the agents in each group is equal to the room rent. The second set of constraints guarantee strong envy-freeness and the third set ensures the individual rationality condition. Theorem 3 ensures that the LP described in Algorithm 1 is

ALGORITHM 1: Strong envy-free allocation with maximum rent

(1) Let $\mathcal{A}$ be a welfare-maximizing allocation

(2) Compute a rent division $\mathcal{R}$ and cost-sharing function $\mathcal{D}$ by the linear program

$$\begin{aligned}
&\max \sum_{k=1}^{n} \sum_{j=1}^{m_k} d_{k,j,\mathcal{A}(k)} \\
&\quad s.t. \\
&\quad \mathcal{R}_k = \sum_{j=1}^{m_i} d_{i,j,k} && \forall i, k \\
&\quad v_{i,j,\mathcal{A}(g_i)} - d_{i,j,\mathcal{A}(g_i)} \geq v_{i,j,k} - d_{i,j,k} && \forall i, j, k \\
&\quad v_{i,j,\mathcal{A}(g_i)} - d_{i,j,\mathcal{A}(g_i)} \geq 0 && \forall i, j
\end{aligned}$$

feasible. However, we still must overcome a technical hurdle: we did not show that the allocation-triple suggested by Algorithm 1 is the one that maximizes the total rent. In fact, by Algorithm 1 we find a solution that maximizes the total rent among the solutions with welfare-maximizing allocation functions. But the allocation-triple with the maximum possible total rent may be obtained by some other allocation functions. Let $\mathcal{S} = (\mathcal{A}, \mathcal{R}, \mathcal{D})$ be the optimal strong envy-free solution and let $\mathcal{A}^*$ be the welfare-maximizing allocation. By strong envy-freeness we know

$$v_{i,j,\mathcal{A}(g_i)} - d_{i,j,\mathcal{A}(g_i)} \geq v_{i,j,k} - d_{i,j,k} \qquad \forall i, j, k. \tag{7}$$

By summing over all agents in group g_i, for all i, k we have:

$$\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \mathcal{R}(\mathcal{A}(g_i)) \geq \sum_{j=1}^{m_i} v_{i,j,k} - \mathcal{R}(h_k). \tag{8}$$

Since Equation (7) holds for all $k \neq \mathcal{A}(g_i)$, it also holds for room $\mathcal{A}^*(g_i)$. Therefore, for all g_i we have:

$$\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \mathcal{R}(\mathcal{A}(g_i)) \geq \sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \mathcal{R}(\mathcal{A}^*(g_i)). \tag{9}$$

Summing Inequality (9) over all the groups yields:

$$\sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \sum_{k=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}(g_i))$$
$$\geq \sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \sum_{k=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}^*(g_i)).$$

Since both $\mathcal{A}$ and $\mathcal{A}^*$ are bijections, we have

$$\sum_{k=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}(g_i)) = \sum_{k=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}^*(g_i)). \tag{10}$$

Hence,

$$\sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} \geq \sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)}. \tag{11}$$

By definition of welfare-maximizing allocation,

$$\sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} \leq \sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)},$$
$$\sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} = \sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)}. \qquad \text{Inequality (11)}$$

Regarding Equation (10),

$$\sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \sum_{i=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}(g_i))$$
$$= \sum_{i=1}^{|\mathcal{G}|}\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \sum_{i=1}^{|\mathcal{G}|} \mathcal{R}(\mathcal{A}^*(g_i)). \tag{12}$$

Equality (12) together with Inequality (9) results in the following expression for all i:

$$\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \mathcal{R}(\mathcal{A}(g_i)) = \sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \mathcal{R}(\mathcal{A}^*(g_i)),$$
$$\sum_{j=1}^{m_i} v_{i,j,\mathcal{A}(g_i)} - \sum_{j=1}^{m_i} d_{i,j,\mathcal{A}(g_i)} = \sum_{j=1}^{m_i} v_{i,j,\mathcal{A}^*(g_i)} - \sum_{j=1}^{m_i} d_{i,j,\mathcal{A}^*(g_i)}. \tag{13}$$

In addition, since Inequality (7) holds for every room k,

$$v_{i,j,\mathcal{A}(g_i)} - d_{i,j,\mathcal{A}(g_i)} \geq v_{i,j,\mathcal{A}^*(g_i)} - d_{i,j,\mathcal{A}^*(g_i)} \qquad \forall i,j. \tag{14}$$

Equation (13) together with Inequality (14) yield:

$$v_{i,j,\mathcal{A}(g_i)} - d_{i,j,\mathcal{A}(g_i)} = v_{i,j,\mathcal{A}^*(g_i)} - d_{i,j,\mathcal{A}^*(g_i)} \qquad \forall i,j.$$

Hence, for all i, j, k, we have

$$v_{i,j,\mathcal{A}^*(g_i)} - d_{i,j,\mathcal{A}^*(g_i)} \geq v_{i,j,k} - d_{i,j,k}.$$

This shows that we can change the allocation of the optimal solution to the welfare-maximizing allocation, without violating the strong envy-freeness condition. Thus, the solution offered by LP maximizes the total rent amongst all admissible allocation-triples.

4 Not Pre-determined Groups

In this section, we consider the case that the groups are not known in advance. As regards the dormitories, for instance, it is more realistic to assume that the students request for a room individually.

For this case, we show that a strong envy-free allocation with the equal cost-sharing policy always exists. Assume that we have a house with n rooms with capacity of m agents per room. In addition, we further suppose that the set of agents is $\mathcal{N} = \{a_1, a_2, \ldots, a_n\}$ and value of i'th room for agent a_j is $v_{i,j}$. The goal is to provide a strong envy free quadruple $\mathcal{S} = (\mathcal{B}, \mathcal{A}, \mathcal{R}, \mathcal{D})$ as defined in Section 2.

Theorem 5. *For the case that the groups are not pre-determined, Strong envy-freeness is consistent with equal cost-sharing.*

Proof. First, we construct an instance of the classic Rent Sharing problem as follows: let $\mathcal{H}' = \{r_{1,1}, \ldots, r_{1,m}, r_{2,1}, \ldots, r_{2,m}, \ldots, r_{n,1}, \ldots, r_{n,m}\}$ be a house consisting m copies of every room in $\mathcal{H}$. Furthermore, let $r_{i,j}$ be the j'th copy of the i'th room in $\mathcal{H}$. Now, we solve the classic Rent Sharing problem instance considering $\mathcal{H}$ and $\mathcal{N}$. First note that for this case Observation 6 holds.

Observation 6 *For every i, j, j', the rent charged for room $r_{i,j}$ is the same as $r_{i,j'}$.*

Observation 6 is because of the fact that the agents in rooms $r_{i,j}$ and $r_{i,j'}$ must not envy each other. Let A^* be the allocation function that allocates a room to each agent and let R^* be the function that assigns a rent to each room. Now, let $\mathcal{S}^* = (\mathcal{B}^*, \mathcal{A}^*, \mathcal{R}^*, \mathcal{D}^*)$ be an allocation quadraple where the cost sharing function $\mathcal{D}^*$ is equal and the rent charged for i'th room of $\mathcal{H}$ is $\sum_{j=1}^{m} r_{i,j}$. Moreover, we define the i'th group of $\mathcal{B}^*$ as the agents located in one of the copies of i'th room and the allocation function $\mathcal{A}^*$ allocates i'th room to g_i.

By observation 6, every agent pays the same rent in $\mathcal{S}^*$ as in the calssic rent sharing instance. Thus, envy-freeness of the classic instance implies strong envy-freeness of $\mathcal{S}^*$. In addition, the agents in the same group pay equal price for their room. Hence, the allocation quadraple $\mathcal{S}^*$ is strong envy-free with the equal cost-sharing policy.

Table 2: Predetermined groups

	Weak Envy-free	Aggregate Envy-free	Strong Envy-free
Equal	✓Obs. 1	✓Lem. 3	× Lem. 1
Proportional	✓Obs. 1	✓Lem. 4	× Lem.2
Free	✓Obs. 1	✓Obs. 1	✓Thm. 3

Table 3: Not pre-determined groups

	Strong Envy-free
Equal	✓Thm. 5
Free	✓Lem. 3, 4

Recall that in Section 3 we proved that when the groups are pre-determined, no allocation-triple can guarantee strong envy-freeness with the equal cost-sharing policy.

5 Conclusion and Future works

In this paper, we considered the Group Rent Sharing problem, which is an extension of the classic Rent Sharing problem to the case where each room must be allocated to a group of agents. We generalized the envy-freeness notion for such situations. We also defined the cost-sharing policy, which adopts the method by which the rent is divided among the resident agents of a room.

We defined three fairness criteria (weak, aggregate, and strong envy-free) and three cost-sharing policies (equal, proportional, and free). Our results encompass several positive and negative results regarding the consistency of different fairness notions and cost-sharing policies. You can find a summary of these results in Tables 2 and 3.

We proposed two positive results regarding strong envy-freeness: consistency of this notion with the free cost-sharing policy in the case that the groups are pre-determined and consistency with equal cost-sharing in the case that the groups are not pre-determined. For both of these cases, we can find the allocation with the maximum total rent. One interesting open question is to give an upper-bound on the ratio of maximum total rent in these two cases. Another direction would be the analysis of the problem in stochastic settings where the valuation of the houses to the agents are drawn from a given distribution.

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