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Main

# Pricing in Population Games with Semi-Rational Agents

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**Abstract**

We consider a pricing game in which two competing sellers offer two similar products on a social network among agents (communities). Each agent chooses, in an iterative manner, between the two products depending on what his neighbours do. This introduces two separate games; one between the agents and one between the two sellers. We show that the first game is a full potential game and provide a polynomial time algorithm to compute where the game converges to. We also study various properties of the second games such as its equilibrium points and its convergence.

*Keywords:* Social Networks, Pricing, Bounded Rationality, Game Theory, Algorithm

## 1. Introduction

How can a seller make profit out of a social network? One reasonable policy for monetizing social networks is to spread the product in a population through the network of individual interactions. Because of the rapid growth and popularity of on-line social networks, the topic has attracted interest among researchers seeking clever policies. For example, several papers have studied agents' behaviours in social markets [1, 2, 3, 4].

Most of these studies focus on a single seller. This work, however, considers two competing sellers and studies various related questions such as the behaviour of buyers, the strategies of the two sellers, and so on. In our model, two companies are competing within a social network. Like the classic approach, the social network is modeled by a graph whose edges represent the interaction between people. The main difference, however, is that the nodes of the graph represent communities in the society rather than individuals. Each community consists of a continuum of potential small agents which interact anonymously. So, the market is modeled based on population games [5].

In our model, the two companies (sellers) announce their prices first and then, agents within communities choose which company to buy from. An agent's utility depends basically on the fraction of neighbours that are buying the same product as that agent. In our model, agents behave cooperatively in a sense that they tend to buy the same product as most of their friends do. Our aim is to study the behaviour of both agents (as consumers) and two competing companies in this game.

To make the setting more realistic, we consider a repeated game in which agents repeatedly revise their decisions. For this, we consider the noisy best-response, logit-response, dynamics for the evolution of the market. In this setting, agents revise their strategies asynchronously. Each agent plays its best-response strategy with some probability close to 1; hence, allowing a slight probability of making mistakes. This may happens in reality when agents' information about the environment are incomplete, when they may make mistakes in their computations, or when agents are not fully rational. The noisy best-response dynamics have been suggested as a method for refining Nash equilibria in games [6, 7, 8, 9, 10].

### *1.1. Our Results*

We consider two separate games in our model. The first one is between agents (buyers) who choose between the two products and the second one is between the two companies that announce their prices and sell their products. For the first game, we show that with the logit-response dynamic, the market always converges to an equilibrium point. We show in Section 3 that the game will be a *full potential game* and its equilibrium point is the global maximum of some potential function. We also prove that agents within the same community buy the same product in the equilibrium. Using this observation, we propose, in Section 5.1, a polynomial-time algorithm for computing the unique equilibrium.

As for the game between the two companies, we study the behaviour of the two companies and obtain several results. We show, in Section 4, that the game has either no pure Nash equilibrium or has a unique one. Furthermore, the best-response dynamic between companies converges to this unique equilibrium. We also prove the existence of such equilibrium for some graph classes such as preferential attachment graphs and regular graphs. We finally present, in Section 5.2, a polynomial-time algorithm for computing the best response strategy for the companies.

### *1.2. Related Work*

In the traditional game theory, we make strong assumptions about knowledge of individuals and consider them fully aware of others. Evolutionary dynamics, on the other hand, are introduced for relaxing these assumptions. Several works (e.g., [6, 7, 8, 9, 10]) have extensively studied these dynamics and pointed out that introducing perturbations to deterministic processes would create distinctive differences in behaviour of dynamics. In a seminal work, Kandori et al. [7] investigate evolutionary noisy best-response dynamics and prove that the dynamics converges to an equilibrium in which all agents adopt the same strategy. This strategy is the one which would be chosen by an agent who has no information about his neighbours or, his neighbours would play completely random. This strategy is named *risk dominant* by Harsanyi and Selten [11].

Blum [8] investigates statistical aspects of various strategy revision protocols. He considers local interaction among a large population of agents in which each agent interacts directly with limited number of agents and interacts indirectly with others through a path in network. He introduces logit-response dynamics and characterizes stationary distribution of related Markov process.

Ellison [9] studied the effect of the underlying graph structure on the game; he specifically discussed convergence time for certain graph classes. Following this work, Montanari et al. [10] studied the logit-response dynamics and made a general and precise connection between the convergence time and the structure of the graph. Our model is inspired by these works with one major difference. Unlike the previous models in which each vertex in these models represents a single agent, vertices in our model correspond to communities. This means that we are not dealing with individuals, rather considering the behaviour of a large groups each containing several individual.

The problem of designing a pricing strategy for a company on a social network is extensively studied in literature (See, e.g., [1, 2, 3, 4]). All these works consider a monopolistic situation in which one single company sells its product and tries to maximize his profit by employing a clever strategy. Hartline et al. [1] and Akhlaghpour et al. [3] assume naive behaviour for consumers. In fact, they study the market with consumers who act myopically and buy the product as soon as they can afford to buy it. They don't make any reasoning about future reaction of their neighbours and their long-term utility. In [1], the seller uses an adaptive price discrimination strategy. In their model, the monopolist visits consumers in some order and offers private prices to each of them; each consumer may accept or reject the offer based on the reactions of her other friends. Akhlaghpour et al. [3] consider a market where the seller iteratively posts a price for the product at several time steps. The price is visible to all buyers at each time step and a buyer may buy the product at a time or wait for a later time. The utility of each buyer depends on the price and the set of her neighbours who have bought the product before her.

In order to consider more intelligent agents, Ahamdipour et al. [2] and Bimpikis et al. [4] model the market as a game. Ahamdipour et al. [2] consider a situation in which different prices are publicly announced over the time. Then the agents choose the time in which they want to buy the product. In these models, agents–who are supposed to be fully rational–strategically choose their best strategy based on full information regarding all future states of the market. They study agents' reactions in Nash equilibrium situation. Bimpikis et al. [4] studied a two stage pricing problem. In this problem the seller first offers the prices for a divisible good and then, the agents simultaneously decide about the amount of purchase. In these studies, agents assumed to be fully rational and do not make mistakes. It seems that the correct model of agents' behaviour probably lies somewhere between these two extremes of myopic agents and fully rational agents.

## 2. Our Model

In our model, we study a society that consists of several large mutually influencing *communities*[1]. Let $n$ be the number of communities and $m^i$ be the mass of people in the $i^{th}$ community. For a subset $T$ of communities, let $m^T = \sum_{i \in T} m^i$ be the mass of people in $T$. Let $m = \sum_i m^i$ be the total mass of the society. We model the interaction between different communities by an undirected graph $G = (V, E)$ whose nodes correspond to communities and an edge $\{i, j\}$ represents an interaction between communities $i$ and $j$. We call this graph the *market graph*. We also allow loops, i.e.

[1]We may use the terms community and population interchangeably.

|  | $A$ | $B$ |
| --- | --- | --- |
| $A$ | $a$ | $c$ |
| $B$ | $d$ | $b$ |

Figure 1: The payoff matrix $\mathbf{U}$

edge $\{i, i\}$, in $G$ to emphasize that agents in a same community influence each others as well. Let $N(i)$ be the set of neighbours of community $i$ including itself. For two subsets $X$ and $Y$ of communities we define $\delta(X, Y) = \sum_{i \in X} \sum_{j \in Y, (i,j) \in E} m^i m^j$ which represents the amount of interaction between the communities in $X$ and $Y$. Note that $X$ and $Y$ may have non-empty intersection. We will also use $\delta(X)$ for $\delta(X, X)$ for simplicity.

Assume there are two products $A$ and $B$ offered by two competing companies with prices $p_A$ and $p_B$, respectively. Each agent chooses either $A$ or $B$; so, its strategy space is the set $S = \{A, B\}$. Let $x_s^i$, where $s \in S$, be the fraction of people in the community $i$ that buy product $s$. Thus, $x_A^i + x_B^i = m^i$ and $x = (x_s^i)$ is a vector of $2 \times n$ elements representing the strategy profile of the game. We define $m_s(x) = \sum_i x_s^i$ to be the mass of population who use product $s \in S$. Let $\mathcal{D}_s^i(x) = \sum_{j \in N(i)} x_s^j$ be the mass of neighbors of community $i$ that use product $s$, for $s \in S$. Also, for every $s \in S$, define $\mathcal{D}_s(x) = \sum_{i \in V} x_s^i \mathcal{D}_s^i(x)$. The utility of every person is obtained by aggregating its utility against every single agent that he interacts with. Let $\mathbf{U}$ (illustrated in Fig. 1) be the payoff matrix for two players. Then, the utility of a person in community $i$ that plays $s$ in a game with strategy profile $x$ would be

$$F_s^i(x) = \mathbf{U}(s, A)\mathcal{D}_A^i(x) + \mathbf{U}(s, B)\mathcal{D}_B^i(x) - p_s \tag{1}$$

We assume in our model that $\mathbf{U}$ is symmetric, i.e. $c = d$ and the game defined by matrix $\mathbf{U}$ is a coordination game, i.e. the players obtain a higher payoff by adopting same strategy. In other words, we have $a > d$ and $b > c$. Without loss of generality and throughout the paper, let $a > b$. Also, for the rest of this paper, we assume that $c = d = 0$; we will prove, in Theorem 1, that this assumption does not hurt the generality of our results.

Our game is in category of *population games* which provide a general framework for studying the strategic interactions in which society consists of several populations. The behaviour of agents in each population are the same. In these games, the number of agents is large, impact of each individual agent is small, and agents interact anonymously, i.e., each agent's payoff depends solely on the distribution of opponents' choices. For more details on population games see [5].

### 2.1. Market Dynamics

As mentioned before, two competing companies are offering products $A$ and $B$ with prices $p_A$ and $p_B$ respectively. In a normal situation, agents update their strategies by looking at their neighbours and buy a product that maximizes their benefit. In our model, we consider *noisy best-response dynamics* in which agents adopt their best response at each iteration with probability close to one. Therefore, there is a slight possibility of making mistakes by agents. More specifically, we study *logit dynamics*. For specific treatment of these dynamics in the context of evolutionary game theory, one can refer to [5].

In our model, the noisy best response dynamics is specified by a parameter $\beta \in R^+$ representing how noisy the system is. In fact, $\beta = \infty$ represents the noise-free or best-response dynamics, and $\beta = 0$ represents the full noisy dynamics in which agents play with no preference. We assume that each agent in a community revises its strategy by arrival of Poisson clock of rate 1. We consider logit-response as revision protocol. So, the probability that an agent in community $i$ takes action $s$ is:

$$P_{i,\beta}(s|x) = \frac{e^{\beta F_s^i(x)}}{\sum_{s' \in S} e^{\beta F_{s'}^i(x)}} \tag{2}$$

This defines a reversible Markov chain with vectors $x$ as its states. As we see later, this game is a *full potential game*, with some potential function $f$. So, the stationary distribution of the Markov chain is

$$\mu(x) \propto e^{\beta f(x)} \tag{3}$$

It is immediate that as $\beta \to \infty$ the dynamic spends most of its time on the global maximum of $f$. We name the global maximum of $f$ the *stationary state* of the market.

We can now prove that assuming $c = d = 0$ does not make any difference in our results.

**Theorem 1.** *Suppose* $w \leq \min(a, b, c, d)$*. Agents' decisions in game defined on matrix* $\mathbf{U}$ *is equivalent to agents' decisions in game with matrix* $\mathbf{U} - w$*, in which* $\mathbf{U} - w$ *is computed by subtracting* $w$ *from all the entries of* $\mathbf{U}$*.*

**Proof :** By equation (1), an agent's payoff in community $i$ for strategy $s$ using the payoff matrix $\mathbf{U} - w$ is

$$\begin{aligned} \hat{F}_s^i(x) &= (\mathbf{U}(s, A) - w)\mathcal{D}_A^i(x) + (\mathbf{U}(s, B) - w)\mathcal{D}_B^i(x) - p_s \\ &= F_s^i(x) - w(\mathcal{D}_A^i(x) + \mathcal{D}_B^i(x)) \end{aligned}$$

It is obvious that both $F$ and $\hat{F}$ result in identical behaviour i.e. give the same probability $P_{i,\beta}(s|x)$ in (2). $\square$

Given $p_A$ and $p_B$, we represent the stationary state of the market by $x(p_A, p_B)$ meaning that the game will eventually converge to the strategy profile $x(p_A, p_B)$. We will later see that $x(p_A, p_B)$ depends solely on the difference of $p_A$ and $p_B$; i. e., if $p_A - p_B = p'_A - p'_B$ then $x(p_A, p_B) = x(p'_A, p'_B)$. We say that profile $(p_A, p_B)$ falls in the region $\mathcal{R}_A^y = \mathcal{R}_B^{m-y}$, if $m_A(x(p_A, p_B)) = y$ and $m_B((p_A, p_B)) = m - y$; i.e., the mass $y$ of the society is using technology $A$ at the stationary state $x(p_A, p_B)$. It is easy to see that increasing $p_A$ decreases $y$ (as depicted in Fig. 2) and since $a > b$, $x(0, 0) \in \mathcal{R}_A^m$.

### *2.2. Market Pricing Game*

Our model introduces a game/competition between the two companies $A$ and $B$. If $x(p_A, p_B) \in \mathcal{R}_A^y = \mathcal{R}_B^{m-y}$ then the utility (profit) of companies $A$ and $B$ are

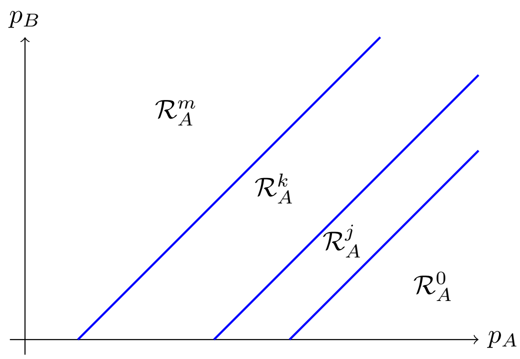


Figure 2: A game with four regions $\mathcal{R}_A^0$, $\mathcal{R}_A^j$, $\mathcal{R}_A^k$, and $\mathcal{R}_A^m$, where $0 < j < k < m$.

$U_A(p_A, p_B) = yp_A$ and $U_B(p_A, p_B) = (m - y)p_B$, respectively. The best response for the company $A$ is the price $p$ which maximize $U_A(p, p_B)$; i. e., $br_A(p_B) = argmax_p U_A(p, p_B)$. Similarly, $br_B(p_A) = argmax_p U_B(p_A, p)$.

In the *Market Pricing Game*, we study the game between the two companies and its properties such as its best response behaviour and existence of equilibria. We also consider the convergence of the best response dynamics of the game.

## 3. Market Behaviour

In this section we analyse the behaviour of communities when the two companies set prices to $p_A$ and $p_B$. This will later help us study the market pricing game. First, we show that our game is a *fully potential game*, as defined in [12], and has various nice properties. So, the maximizer of potential function will characterize the market stationary state when $\beta \to \infty$. We then use this property to find the market stationary state. We show that the stationary state is very simple when $p_A \leq p_B$. In this case, in the stationary state all agents playing strategy $A$. But the problem is not trivial when $p_A > p_B$. In this case, we design a polynomial time algorithm that characterizes the stationary state of the market.

### *3.1. Full Potential Games*

Our main result of this section is that our game is a fully potential game. We use the following definition fro [12]. For more details and useful intuitions, refer to the main article.

**Definition 1.** *Let $F : \mathbb{R}_+^n \to \mathbb{R}^n$ represent a population game. We call $F$ a* ***full potential game*** *if there exist a continuously differentiable function $f : \mathbb{R}_+^n \to \mathbb{R}$ satisfying*

$$\nabla f(x) = F(x), \forall x \in \mathbb{R}_+^n \tag{4}$$

In potential games we can capture all information about agents incentives in a scalar valued function, called *potential function*. Existence of such function provides us with many nice properties and enable us to derive various results about our model. In our model, the function $F$ takes a vector $x$ of $2n$ values ($x_s^i$'s) and output the utilities, i.e., the vector of $F_s^i$'s. We prove that our game is full potential by simply finding an $f$ that satisfies equation (4).

**Theorem 2.** *The function $f$ defined below is the potential function for the game $F$ defined on graph $G = (V, E)$ with payoff matrix $\mathbf{U}$:*

$$f(x) = \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \tag{5}$$

**Proof :** We have

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\sum_{i\in V}\sum_{j\in N(i)} a x_A^i x_A^j + \sum_{i\in V}\sum_{j\in N(i)} b x_B^i x_B^j\right) \\ &- p_A \sum_{i\in V} x_A^i - p_B \sum_{i\in V} x_B^i \end{aligned}$$

Note that, as mentioned before, $N(i)$ includes $i$ itself. The partial derivative of $f$ with respect to arbitrary $x_A^i$ is

$$\frac{\partial f(x)}{\partial x_A^i} = \frac{1}{2}\left(2\sum_{j\in N(i)} a x_A^j\right) - p_A = a\mathcal{D}_A^i(x) - p_A = F_A^i(x)$$

Comparing with (1) the proof is complete. □

*3.2. Market Stationary State*

In this section we study the stationary state of the market. First we provide a lemma that relates global maximum of potential function to the stationary state of the market. Then we characterize the global maximum of potential function $f$ for the case that $p_A \le p_B$. Finally, We will study the case $p_A > p_B$ which is more complicated.

As stated before, we consider noisy best response dynamics. So, strategies are updated with respect to probability in (2). In this case, potential games have a nice property described in the following lemma. It is worth mentioning that this lemma is a succinct result of a more detailed description in [5], where the number of agents in each population is infinite.

**Lemma 3.** *Let $F$ be a potential game with potential function $f$. Then invariant distribution of Markov chain defined on logit-response dynamics is $\mu(x) \propto e^{\beta f(x)}$.*

When $\beta \to \infty$ the dynamic converges to the global maximum of $f$. Speaking more precisely, the dynamic spends most of its time around the global maximum of $f$. So, finding the global maximum of $f$, market stationary state, is important to estimate the outcome of the game.

First, we show in Proposition 4 that the stationary state is the state of all agents playing strategy $A$, when $p_A \le p_B$.

**Proposition 4.** *The logit-response dynamic will converge to the state of all agents playing strategy $A$, if $p_A \leq p_B$ and $\beta \to \infty$.*

**Proof :** Let $y$ be the state of all agents playing strategy $A$ and $x$ be any other state. By equation (5), we can rewrite $f(y)$ as:

$$f(y) = \frac{1}{2}a\mathcal{D}_A(y) - p_A m_A(y) = \frac{1}{2}a \sum_{i \in V} \sum_{j \in N(i)} m^i m^j - p_A m$$

We bound $f(x)$ as follows:

$$\begin{aligned} f(x) &= \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \\ &\leq \frac{1}{2}\left(a\mathcal{D}_A(x) + a\mathcal{D}_B(x)\right) - p_A m_A(x) - p_A m_B(x) \\ &= \frac{1}{2}a \sum_{i \in V} \sum_{j \in N(i)} (x^i_A x^j_A + x^i_B x^j_B) - p_A m \end{aligned}$$

Now by knowing that $x^i_A x^j_A + x^i_B x^j_B \leq (x^i_A + x^i_B)(x^j_A + x^j_B) = m^i m^j$, we can conclude $f(x) \leq f(y)$. So, the maximum of $f$ happens at $y$ and, therefore, the dynamic converges to the state of all agents playing $A$ by Lemma 3. $\square$

Computing the stationary state is more complicated when $p_A > p_B$. In order to solve the problem in this case, we observe, in Lemma 5, that in the long run each community will be *homogeneous*, i.e. all people within same community buy same product. This fact helps us to predict the stationary state of the market in polynomial time in Theorem 6. The idea is to build a weighted graph whose minimum cut characterizes the stationary state. The proofs are omitted here and appear separately in Section 5.1

**Lemma 5.** *In the logit-response dynamic each community will be homogeneous in the long run, when $\beta \to \infty$.*

**Theorem 6.** *We can predict the stationary state of the market in polynomial time in the logit response dynamic, when $\beta \to \infty$.*

### 4. Market Pricing Game

In this section we study the game between the two competing companies. First, using the results of previous section, we resolve the best-response price for each company. We show that the game has either no pure Nash equilibrium or has a unique one in which $p_B = 0$. Then we consider the best-response dynamic and prove that if the game has a pure Nash equilibrium then the best-response dynamics will converge to it.

It is important to point out here, for the market pricing game, we can compute each player's best response in reasonable amount of time. In Section 5.2 we have introduced a polynomial time algorithm in number of communities, in which each company knowing its opponent's price, can compute the most profitable response.

It is worth mentioning that, in this setting, we can model monopolistic societies by just setting $b$ and the price of product $B$ to zero. So it is just one company in the market who should decide the best price for its product.

Given the results of previous sections, the game between the companies could be simplified as follows. Two companies announce two prices $p_A$ and $p_B$. The maximum of $f$ is computed. As stated in Lemma 5, every community would be homogeneous in the long run. Let $S_A$ be the set of communities who buy $A$ and $S_B = V - S_A$ be those who buy $B$. The utilities of the two communities are $p_A m^{S_A}$ and $p_B m^{S_B}$, respectively.

### *4.1. Pure Nash Equilibrium*

We now study equilibrium aspects of pricing game. In homogeneous state of the market, we can write $f$ as follows.

$$
\begin{aligned}
f(x) &= \frac{1}{2}(a\delta(S_A) + b\delta(S_B)) - p_A m^{S_A} - p_B m^{S_B} \\
&= f_\delta - f_v - \mathcal{C}
\end{aligned}
$$

where $f_\delta = \frac{1}{2}(a\delta(S_A) + b\delta(S_B))$, $f_v = (p_A - p_B)m^{S_A}$ and $\mathcal{C} = p_B m$. Since $\mathcal{C}$ is a constant independent of $S_A$, maximizing $f$ is equivalent to maximizing $f_\delta - f_v$. Note that $f_\delta$ is independent of $p_A$ and $p_B$, and solely depends on the structure of the graph. Let $f_\delta^y = \max_{m^{S_A}=y} f_\delta$. Assume $f_\delta^y = 0$, if there is no set $S_A$ with $m^{S_A} = y$. Therefore, when $p_A$ and $p_B$ is fixed, maximizing $f$ is equivalent to finding $y$ that maximizes $f_\delta^y - y\alpha$, in which $\alpha = p_A - p_B$.

Let $(p_A, p_B)$ be a strategy profile of the pricing game. When $\alpha = 0$ then by Proposition 4 all communities adopt $A$ and, hence, $S_A = m$. As $\alpha$ increases, less communities buy $A$. Let $\alpha_{n_i}$ be the very first point that when $\alpha = \alpha_{n_i}$ then the mass of communities that buy $A$ changes to some new value $n_i$. Let the set of threshold points be $\alpha_{n_1} < \alpha_{n_2} < \cdots < \alpha_{n_k}$. For convenience we add $\alpha_{n_0} = 0$. It is clear that $m = n_0 > n_1 > \cdots > n_k = 0$. So, when $p_A - p_B \in [\alpha_{n_j}, \alpha_{n_{j+1}})$ then $m^{S_A} = n_j$ and the utility of company $A$ is $n_i p_A$. See Fig. 3 for illustration.

**Lemma 7.** *If $(p_A, p_B)$ be a Nash equilibrium then, $\alpha$ is slightly less than $\alpha_{n_1}$ and $p_B = 0$.*

**Proof :** If $\alpha_{n_j-1} < \alpha < \alpha_{n_j}$ for some $j \geq 1$ then $B$ increases his price until $\alpha = \alpha_{n_j}$ (See Fig. 3). This increases $B$'s payoff as it will not affect the communities that buy $B$. If $\alpha = \alpha_{n_j}$ for some $1 \leq j < n_k$ then $A$ increases his price until it is slightly less than $\alpha_{n_{j+1}}$. This increases $A$'s payoff as it will not affect the communities that buy $B$. If $\alpha = \alpha_{n_k}$, i.e. no one buys $A$, then $A$ can decrease his price until at least one community buys $A$ and brings more utility to $A$. So, we must have $\alpha < \alpha_{n_1}$. If the strategy domain of companies is continuous then we don't have any Nash equilibrium as company $A$ wants to make $\alpha$ as close to $\alpha_{n_1}$ as possible which gives no Nash equilibrium. But, if we discretized the strategy domain then the only possible $\alpha$ is the largest value (in the discrete domain) less than $\alpha_{n_1}$. Let this value be $\alpha_{n_1}^-$. We argue that $p_B = 0$ as if not $B$ can decrease its price to $0$ and the new $\alpha$ would be at least $\alpha_{n_1}$ which means some communities buy $B$ and $B$ gets more utility. $\square$

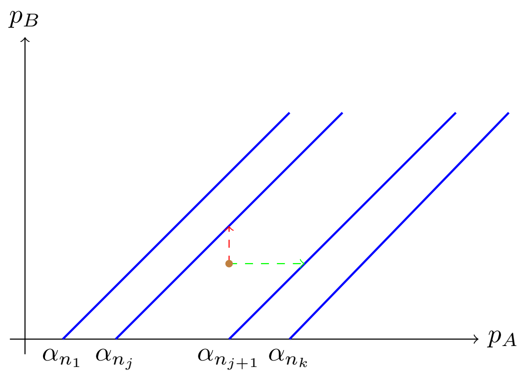


Figure 3: The action of company $A$ ($B$) has been shown by green(red) line.

So, the only possible Nash equilibrium is $(\alpha_{n_1}^-, 0)$. At this point, $B$ is obviously playing best response as he does not get any utility no matter how he plays. However, $A$ necessarily is not playing best response as he may gain more profit by increasing his price.

**Theorem 8.** *If the strategy domain of companies is continuous then we have no Nash equilibrium. Otherwise,* $(\alpha_{n_1}^-, 0)$ *is the unique Nash equilibrium if and only if* $\alpha_{n_1}^- \in br_A(0)$.

*4.2. Market equilibrium on special graphs*

In this section we show that pure Nash equilibrium exists for some special graphs such as regular and preferential attachment graphs.

We first obtain the following sufficient condition for having a Nash equilibrium and then prove it for the above class of graphs. Recall that $f_\delta^y = \max_{m^{S_A=y}} f_\delta$.

**Lemma 9.** *If* $mf_\delta^y < yf_\delta^m + (m-y)f_\delta^0$ *for every* $y < m$*, then* $br_A(0) = \alpha_1^-$ *and the market has a unique equilibrium*

**Proof :** We prove that under the above conditions there is only one single threshold point, i.e., as $\alpha$ increases the situation changes from *all playing A* to *all playing B*. Let $\alpha$ be a point at which $|S_A| = y$ in the maximum of $f$. At this point we have $f_\delta^y - y\alpha \geq f_\delta^0 - 0 \times \alpha = f_\delta^0$ and $f_\delta^y - y\alpha \geq f_\delta^m - m\alpha$. So, we have $\frac{f_\delta^m - f_\delta^y}{m-y} \leq \alpha \leq \frac{f_\delta^y - f_\delta^0}{y}$ which means $mf_\delta^y \geq yf_\delta^m + (m-y)f_\delta^0$; this contradicts the lemma condition. Therefore, either all or no communities buy $A$. Obviously, $A$'s best response at this situation is to play $\alpha_1^-$. So we have a unique Nash equilibrium by Theorem 8. $\square$

We conclude this section by showing that several real world market graphs satisfy the condition of Lemma 9 and have pure Nash equilibrium. For the theorem below we consider *uniform markets*, in which we assume the that all populations masses are similar i.e. we have a uniform distribution of agents among populations. In fact, we assume there are $n$ communities in the market with $m_i = 1$. So the total mass of society is $m = n$. This game is important when we want to focus on the structure of the market graph.

**Theorem 10.** *For the uniform markets, if market graph is a regular or preferential attachment then it has a Nash equilibrium.*

**Proof :** It suffices to prove the condition of Lemma 9.

**Regular graph**: Assume we have a regular graph of degree $d$ with $e = nd/2 = md/2$ edges. Note that $f_\delta^m = ae$, $f_\delta^0 = be$ and $f_\delta^y < (ady + bd(m-y)/2$. So $mf_\delta^y < md/2(ay + b(m-y)) = e(ay + b(m-y)) = yf_\delta^m + (m-y)f_\delta^0$

**Preferential Attachment Graphs:** Assume we have a preferential attachment graph with parameter $d$ with $e = nd = md$ edges. In this model each new node creates exactly $d$ edges to the previous nodes. Note that $f_\delta^m = amd$ and $f_\delta^0 = bmd$. On the other hands, consider an induced sub-graph $G'$ with $y$ vertices. Note $G'$ is connected to the $G - G'$ with at least one edge. So, $G'$ has less than $yd$ edges. Therefore $f_\delta^y < ayd + b(m-y)d$, which implies $mf_\delta^y < yf_\delta^m + (m-y)f_\delta^0$.

□

## 5. Algorithmic Aspects

In this section we propose polynomial time algorithms for two problems. First, we consider the problem of computing the stationary state in Section 5.1. The main result is the proof of Lemma 5 and Theorem 6 that we have stated in Section 3. Second, we propose a polynomial time algorithm for computing best response for companies in the market pricing game in Section 5.2.

### *5.1. Computing the Stationary State*

Let $p_A$ and $p_B$ be fixed. As we know from Lemma 3, the market converges to the maximum of the potential function $f$. Note that, we have shown in Proposition 4 that in the stationary state, all agents will play strategy $A$, when $p_A \leq p_B$. So, we focus on the case $p_A > p_B$ and propose a polynomial-time algorithm to compute such a maximum. Our solution is based on an algorithm for the *Maximum Weighted Set Problem*. This problem has been defined below.

**Definition 2. Maximum Weighted Set Problem (MWSP):** *we are given a directed grpah $G = (V, E)$ with (possibly negative) weights $I_i$ on vertices, and non-negative weights $w_{ij}$ on edges. The aim is to find a subset $S \subseteq V$ so as to maximize*

$$W_S = \sum_{i \in S} I_i + \sum_{\substack{(i,j) \in E \\ i,j \in S}} w_{ij} \tag{6}$$

**Lemma 11.** *The MWSP can be solved in polynomial time.*

**Proof :** The idea is to build a weighted graph whose minimum cut is the solution to the MWSP. For every node $i$, let $h_i = I_i + \sum_{j \in N(i)} w_{ij}$. We build a graph $G'$ out of $G$ as follows. Add two new nodes $s$ and $t$. For every $i$ with $h_i < 0$ add an edge with weight $-h_i$ from $i$ to $t$. For every vertex $i$ with $h_i \geq 0$ add an edge from

$s$ to $i$ of weight $h_i$. The value of the out-cut from any set $S$ which contains $s$ is: $\partial^+(S) = \sum_{\substack{h_i>0\\ i\in T}} h_i + \sum_{\substack{h_i<0\\ i\in S}} -h_i + \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}$, where $T = V(G') - S$. Let $W = \sum_{h_i>0} h_i = \sum_{\substack{h_i>0\\ i\in S}} h_i + \sum_{\substack{h_i>0\\ i\in T}} h_i$. We can rewrite $W - \partial^+(S)$ as:

$$\begin{aligned} W - \partial^+(S) &= \sum_{\substack{h_i>0\\ i\in S}} h_i + \sum_{\substack{h_i<0\\ i\in S}} h_i - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij} \\ &= \sum_{i\in S} h_i - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij} \\ &= \sum_{i\in S}(I_i + \sum_{j\in N(i)} w_{ij}) - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij} \\ &= \sum_{i\in S} I_i + \sum_{\substack{(i,j)\in E\\ i,j\in S}} w_{ij} \end{aligned}$$

Since $W$ is a constant independent of $S$, we conclude that maximizing $\sum_{i\in S} I_i + \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}$ is equivalent to minimizing $\partial^+(S)$ which could be done in polynomial time. $\square$

Lemma 5 helps us to find a connection between MWSP and computing the stationary state. So, we first prove this lemma. Then, we use the algorithm for the MWSP and compute the stationary state of the market and prove Theorem 6.

**Proof of Lemma 5:** Fix community $i$. As we saw in the proof of Lemma 3, when $\beta \to \infty$, the dynamic converges to the global maximum of $f$. The part of $f$ that depends on population $i$ (i.e. involves $x_A^i$ and $x_B^i$) is:

$$\begin{aligned} g(x_A^i) &= \frac{1}{2}\left(a x_A^i x_A^i + b x_B^i x_B^i\right) + x_A^i \sum_{\substack{j\in N(i)\\ i\neq j}} a x_A^j + x_B^i \sum_{\substack{j\in N(i)\\ i\neq j}} b x_B^j \\ &\quad - p_A x_A^i - p_B x_B^i \end{aligned}$$

Since $x_B^i = m_i - x_A^i$, $g(x_A^i)$ will be quadratic in $x_A^i$ and the coefficient of ${x_A^i}^2$ is $C = \frac{1}{2}(a+b) > 0$. Therefore, $g(x_A^i)$ takes its maximum on extreme points, i.e. $x_A^i = 0$ or $x_A^i = m^i$. Since $x^i$'s are independent, the maximum of $f$ happens when for every $i$, $x_A^i = 0$ or $x_A^i = m^i$. $\square$

**Proof of Theorem 6:** We know from Lemma 5 that each population is homogeneous, so it is suffices to find each population's strategy. We reduce this problem to the MWSP as follows. As proven before, the dynamic of the game converges to the global maximum of the potential function $f$. Let $S_A$ and $S_B$ be the set of communities in $G$ that play $A$ and $B$, respectively. We can write potential function (5) for this state of the game as below:

$$f = \frac{1}{2}(a\delta(S_A) + b\delta(S_B)) - p_A m^{S_A} - p_B m^{S_B} \tag{7}$$

By replacing $m^{S_B}$ by $m - m^{S_A}$ and $\delta(S_B)$ by $\delta(V) - \delta(S_B, S_A) - \delta(S_A, S_B) - \delta(S_A)$ we have:

$$
\begin{aligned}
f &= \frac{1}{2}(a\delta(S_A) + b\delta(V) - b\delta(S_B, S_A) - b\delta(S_A, S_B) - b\delta(S_A)) \\
&\quad -p_A m^{S_A} + m^{S_B} - p_B m
\end{aligned}
$$

By omitting constant terms that do not affect the maximization, the problem reduces to the following:

$$
\max_{S_A} f = \frac{1}{2}((a-b)\delta(S_A) - b\delta(S_A, V - S_A) - b\delta(V - S_A, S_A)) + (p_B - p_A)m^{S_A} \quad (8)
$$

We show that the above value is the solution to the MWSP on some graphs $G_W$ that is constructed from $G$ as follows. The vertex set of $G_W$ is that of $G$. The weight $I_i$ of every vertex $i$ is $(p_B - p_A)m^i - bm^i \sum_{j \in N(i)} m^j$ and $w_{ij}$, for every edge $(i, j)$ is $\frac{1}{2}(a+b)m^i m^j$.

For every set $S \subseteq V$ we have

$$
\begin{aligned}
W_S &= \sum_{i \in S} \left( (p_B - p_A)m^i - bm^i \sum_{j \in N(i)} m^j \right) + \sum_{\substack{(i,j) \in E \\ i,j \in S}} \left( \frac{1}{2}(a+b)m^i m^j \right) \\
&= (p_B - p_A)m^S - b\delta(S, V) + \frac{1}{2}(a+b)\delta(S) \\
&= \frac{1}{2}((a-b)\delta(S) - b\delta(S, V - S) - b\delta(V - S, S)) + (p_B - p_A)m^S
\end{aligned}
$$

It is clear that finding a maximum weighted set in $G_W$ is equivalent to finding a set $S_A$ that maximizes (8) and, hence, maximizes the potential function $f$. □

*5.2. Best-response Pricing*

An interesting and important question that we can resolve is the best response strategy of companies in the market pricing game. Given the price of company $B$, $p_B$, what price $p_A$ should the company $A$ set so as to benefit most?

Let us fix $p_B$. We first obtain lower and upper bounds for the best response of $A$ and then compute it by using binary search. We know from Proposition 4 that if $p_A \leq p_B$ then all populations will play $A$. So the minimum of $p_A$ is obviously $p_B$. Also the maximum of $p_A$ is the point where no one play $A$. The following lemma characterizes this point.

**Lemma 12.** *Global maximum of potential function $f$ is the state of all agents playing strategy $B$, if for all $i \in V$ we have $p_A > p_B + \frac{1}{2}(a-b)\sum_{j \in N(i)} m^j$. So the maximum of $p_A$ is at most $p_A^{\max} = p_B + \max_i(\frac{1}{2}(a-b)\sum_{j \in N(i)} m^j)$.*

**Proof :** Let $y$ be the state of all agents playing strategy $B$, and $x$ be an arbitrary state. We have:

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \\
&\leq \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_B m - \frac{1}{2}a\sum_{i\in V}\sum_{i\in N(i)} x_A^i m^j + \frac{1}{2}b\sum_{i\in V}\sum_{i\in N(i)} x_A^i m^j \\
&= \frac{1}{2}a\sum_{i\in V}\sum_{j\in N(i)}(x_A^i x_A^j - x_A^i m^j) + \frac{1}{2}b\sum_{i\in V}\sum_{j\in N(i)}(x_B^i x_B^j + x_A^i m^j) - p_B m \\
&\leq \frac{1}{2}b\sum_{i\in V}\sum_{j\in N(i)} m^i m^j - p_B m = f(y)
\end{aligned}$$

$\square$

It is clear that the total mass of communities that play $A$ decreases as $p_A$ increases. Also, we know from Lemma 5 that each community is homogeneous. So there are certain points at which if we increase $p_A$ a little more, at least one population will change its strategy. We call these points as *threshold points*. The following lemma proves that by increasing $p_A$ no population will change its strategy from $B$ to $A$.

**Lemma 13.** *Let $S_A$ and $S'_A$ be the set of communities that play $A$ in the stationary state of the market when the price of company $B$ is $p_B$ and the price of company $A$ is $p_A$ and $p'_A > p_A$, respectively. Then $S'_A \subseteq S_A$.*

**Proof :** Assume $S'_A \nsubseteq S_A$. Note that the set of communities $S_A$ play $A$ in the the stationary state of the market with prices $p_A$ and $p_B$. Using proof arguments of Theorem 6, we can conclude that $S_A$ is the maximum weighted set of graph $G_W$ with $I_i = (p_B - p_A)m^i - bm^i \sum_{j\in N(i)} m^j$ and $w_{ij} = \frac{1}{2}(a+b)m^i m^j$. So the weight of set $S_A$ is greater than or equal to the weight of set $S_A \cup S'_A$, which means:

$$\begin{aligned}
\sum_{i\in S_A} I_i + \sum_{\substack{(i,j)\in E \\ i,j\in S_A}} w_{ij} &\geq \sum_{i\in S_A\cup S'_A} I_i + \sum_{\substack{(i,j)\in E \\ i,j\in S_A\cup S'_A}} w_{ij} \\
\Rightarrow 0 &\geq \sum_{i\in S'_A - S_A} I_i + \sum_{\substack{(i,j)\in E \\ i\in S'_A - S_A, j\in S_A\cup S'_A}} w_{ij} \qquad (9)
\end{aligned}$$

Similarly, we can show that $S'_A$ is the maximum weighted set of graph $G_{W'}$ with $I'_i = I_i - (p'_A - p_A)m^i$ and $w'_{ij} = w_{ij}$. So the weight of set $S'_A$ is greater than or equal to the weight of set $S_A \cap S'_A$, which means:

$$\begin{aligned}
\sum_{i\in S_A\cap S'_A} I'_i + \sum_{\substack{(i,j)\in E \\ i,j\in S_A\cap S'_A}} w'_{ij} &\leq \sum_{i\in S'_A} I'_i + \sum_{\substack{(i,j)\in E \\ i,j\in S'_A}} w'_{ij} \\
\Rightarrow 0 &\leq \sum_{i\in S'_A - S_A} I'_i + \sum_{\substack{(i,j)\in E \\ i\in S'_A - S_A, j\in S'_A}} w'_{ij} \qquad (10)
\end{aligned}$$

Because $p'_A > p_A$, we have $I'_i < I_i$, for every $q \leq i \leq n$. On the other hands, we assumed $S'_A \not\subseteq S_A$, which means $|S'_A - S_A| > 0$. So $\sum_{i \in S'_A - S_A} I'_i < \sum_{i \in S'_A - S_A} I_i$. Now using inequalities (9, 10) and the fact $w'_{ij} = w_{ij}$, we conclude $\sum_{\substack{(i,j) \in E \\ i \in S'_A - S_A, j \in S_A - S'_A}} w_{ij}$ is less than zero. This is a contradiction because we know $w_{ij} \geq 0$, for every $0 \leq i, j \leq n$. $\square$

So, one can fix $y$, as the total mass of populations who buy $A$ and compute the maximum possible value of $p_A$ for which at least mass of $y$ people buy $A$. The latter could be done by a simple binary search algorithm. This gives a profit of at least $yp_A$. Finally, we find this maximum over all values of $y$ and take the maximum.

Note that for each pricing Theorem 6 finds each population's strategy in polynomial time. So if we accept $\epsilon$ deviation, we can find each threshold point in $O(n^3 \log \frac{(p_A^{\max} - p_B)}{\epsilon})$ time. In which $O(n^3)$ is for finding minimum-cut, in order to find each population's strategy, as described in proof of Theorem 6. As mentioned above we should take the maximum over all threshold points. Using Lemma 13, we can conclude that number of these points at most would be equal to number of communities. Hence, we have the following theorem.

**Theorem 14.** *In the market pricing game each company, knowing its opponent product price, can determine the best price in polynomial time in number of communities.*

## 6. Conclusion

We considered a network of communities through which two sellers compete on selling a similar product. We analyzed both games (between communities and between sellers) and obtained several results regarding the equilibria of the game, the convergence problem in the dynamic market and their efficient computation.

One important research direction is to study convergence rates in the above settings: how long does it take to get to or close to the convergence point? Are there some general graph classes for which there exist rapid convergence?

Another interesting problem is to consider more than two sellers. This seems like a challenging but very interesting question. Also, a similar problem is to different treatments of communities. For example, a seller be able to offer different prices to different communities.

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