Efficient Computation of Query Point Visibility in Polygons with Holes

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Abstract

In this paper, we consider the problem of computing the visibility polygon of a query point inside polygons with holes. The goal is to perform this computation efficiently per query with more cost in the preprocessing phase. Our algorithm is based on solutions in [12] and [13] proposed for simple polygons. In our solution, the preprocessing is done in time $O(n^3 \log(n))$ to construct a data structure of size $O(n^3)$. It is then possible to report the visibility polygon of any query point q in time $O((1+h')\log n + |V(q)|)$, in which n and h are the number of the vertices and holes of the polygon respectively, $|V(q)|$ is the size of the visibility polygon of q, and h' is an output and preprocessing sensitive parameter of at most $\min(h, |V(q)|)$. This is claimed to be the best query-time result on this problem so far.

Keywords: *visibility polygon, visibility decomposition, polygon with holes*

1 Introduction

Two points inside a polygon are visible from each other if their connecting segment remains completely inside the polygon. The visibility polygon of a point q, called $V(q)$, in a polygon $\mathcal{P}$ is defined as the set of points in $\mathcal{P}$ that are visible from q. The problem of finding $V(q)$ of a query point q has been considered for two decades. For simple polygons, linear time optimal algorithms have been proposed [1, 8, 3, 11]. For polygons with holes, the worst case optimal algorithms with total time of $O(n \log n)$ were presented in [9] and [7]. This was later improved to $O(n + h \log h)$ in [5].

This problem in the query version has been considered by few. The visibility complex and visibility decomposition are two methods of performing the preprocessing step. Visibility complex was first presented in [6]. In this method, $V(q)$ of any query point q can be reported in time $O(|V(q)| \log n)$ by $O(n \log n)$ preprocessing time and using $O(n)$ space. This is the best result so far for polygon with holes. In comparison, our method uses more preprocessing time and space, but the query time is done more efficiently in most cases.

The notion of visibility decomposition is to decompose the polygon into the "visibility regions" so that all points in a single such region have equivalent visibility polygons. Two visibility polygons are equivalent if they are composed of the same sequence of vertices and edges from the underlying polygon. In such a decomposition, the visibility regions are determined in the preprocessing phase and their visibility polygons are computed and maintained in a proper data structure. For any point q, $V(q)$ can then be obtained by refining the visibility polygon of the region that contains q.

In a simple polygon with n vertices, $V(q)$ can be reported in time $O(\log n + |V(q)|)$ by spending $O(n^3 \log n)$ of preprocessing time and $O(n^3)$ space [10, 12]. Another improvement to this method was done in [13] where the preprocessing time and space were reduced to $O(n^2 \log n)$ and $O(n^2)$ respectively, at expense of more query time of $O(\log^2 n + |V(q)|)$.

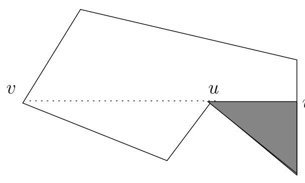

Figure 1: uu' as a window of a polygon.

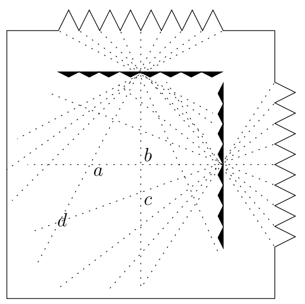

Figure 2: A polygon with $O(n^4)$ v-regions and *sinks*.

In this paper we apply the visibility decomposition method on non-simple polygons and present a method to extend the above algorithms for such polygons. In an overall view, we add some new edges and vertices to the non-simple polygon (called *cut-diagonal*) so that the polygon can be unfolded along these diagonals and converted to a simple polygon. Then, we use an existing algorithm on simple polygons (one of [10] and [12]) to compute a preliminary version of the $V(q)$. With some additional work, we find the final $V(q)$.

For a polygon of total n vertices and h holes, our algorithm needs the preprocessing time of $O(n^3 \log n)$ and memory of size $O(n^3)$. Any query can then be handled in time $O((1+h')\log(n) + |V(q)|)$ in which h' is an output and preprocessing sensitive parameter of at most $\min(h, |V(q)|)$.

In the rest of this paper and in section two, the visibility decomposition will be applied to non-simple polygons and its time and space complexities will be analyzed. In section three, the new algorithm will be presented. Finally, the materials will be summarized and concluded in section four.

2 Visibility Decomposition

Let $\mathcal{P}$ be a polygon with h holes $H_1, H_2, \cdots, H_h$. Also let q be the query point for which the visibility polygon $V(q)$ is to be computed. A visibility decomposition of $\mathcal{P}$, denoted as v-decomposition($\mathcal{P}$), is to partition $\mathcal{P}$ into a set of smaller visibility regions R, called v-regions, such that for each region $A \in R$, the same sequence of vertices and edges of $\mathcal{P}$, called A's *visibility sequence* and denoted by v-sequence(A), are visible from any point in A.

To construct a v-decomposition($\mathcal{P}$), we first identify the boundary segments of its regions R. We then construct a subdivision from these segments. The boundary segments are either the edges of $\mathcal{P}$, or segments that we call *windows* of $\mathcal{P}$. As shown in Figure 1, a window uu' is an extension of the segment between two mutually visible vertices u and v of $\mathcal{P}$. This is because the points below the window uu' are not visible from vertex v, while the upper points are. It is easy to prove that no other kinds of segments are involved in construction of the v-decomposition. More details on the properties of such a decomposition can be found in [12] and [13].

Lemma 1 . *The number of the v-regions of a non-simple polygon $\mathcal{P}$ is $O(n^4)$ and this bound is tight.*

Proof. Each vertex of $\mathcal{P}$ can be an endpoint of at most n different windows. Hence, the number of all windows is $O(n^2)$. Any two windows can intersect which will lead to at most $O(n^4)$ v-regions. This bound is tight as shown in Figure 2. □

It is easy to see that the v-sequences of two adjacent v-regions in a simple polygon differ only in a single vertex which is visible from the points of one region and is invisible from the other. This fact helps reduce the space complexity of maintaining the v-sequences of the v-regions in simple polygons. This is done by defining the *sink regions*. A region is sink if the size of its v-sequence is smaller compared to all of its adjacent regions. It is sufficient to only maintain the v-sequences of the sinks, from which the v-sequences of all other regions can be computed.

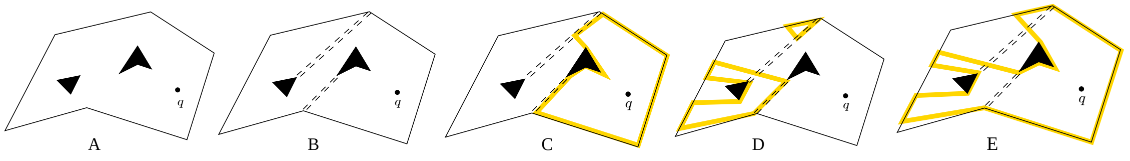

Figure 3: Computing $V(q)$ inside a non-simple polygon: A) The original polygon $\mathcal{P}$, B) The cut-diagonals to produce a simple polygon $\mathcal{P}_s$, C) The visibility polygon $V_s(q)$ targeted to $\mathcal{P}_s$, D) Extra segments of $\mathcal{P}$ viewed from q through the cut-diagonals, and E) the final $V(q)$ in $\mathcal{P}$

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A directed dual graph is built on the v-regions [10, 12]. In a simple polygon, there are $O(n^2)$ sinks. This reduces the space requirement of v-decomposition of simple polygons to $O(n^3)$. Unfortunately, the above property does not hold for non-simple polygons.

Lemma 2 *.The space complexity of maintaining the v-sequences of the regions in a non-simple polygon is $O(n^5)$.*

Proof. This bound is trivially true, because the number of v-regions is $O(n^4)$ and any one of these regions can have a v-sequence of size $O(n)$. On the other hand, Figure 2 shows a polygon with $O(n^4)$ sink regions each with v-sequences of size $O(n)$. This is because any one of the v-regions like *abcd* contains at least one sink. □

By computing the v-regions R of $\mathcal{P}$ and maintaining their v-sequences, $V(q)$ of an arbitrary point q inside $\mathcal{P}$ can be answered as follows: A point location structure will be built over the v-regions. From this, the region $r(q)$ containing q can be found in time $O(\lg n)$. The v-sequence($r(q)$)is then traced and refined to exactly compute $V(q)$. This refinement takes linear time in terms of the size of $V(q)$ [12]. Therefore,

Theorem 1 *. Using $O(n^5 \log n)$ time to preprocess a polygon $\mathcal{P}$ and maintaining a data structure of size $O(n^5)$, it is possible to report $V(q)$ in time $O(\lg n + |V(q)|)$.*

Proof. We first compute the $V(r)$ for each vertex r of $\mathcal{P}$ [7]. All windows can then be found in time $O(n^2 \log n)$. The v-decomposition and its dual graph can then be constructed in time $O(n^4 \log n)$ [14]. The point location structure on the v-decomposition can be constructed in time $O(n^4 \log n)$ [15, 16, 17, 18]. Since there can be $O(n^4)$ sinks, computing the v-sequences takes $O(n^5 \log n)$ time and the size of anyone of these v-sequences can be $O(n)$. Hence, the total preprocessing time is $O(n^5 \log n)$ and the size of the required data structure is $O(n^5)$. $V(q)$ is then found in time $O(\log n + |V(q)|)$ as describe above. □

3 The Proposed Algorithm

Clearly, time and space complexities of the previous algorithm is too high and it is not acceptable in all applications. In this section we propose another algorithm for this problem that needs less preprocessing time and space at expense of higher cost of query time.

The first step of this algorithm is to convert the initial non-simple polygon $\mathcal{P}$ into a simple polygon $\mathcal{P}_s$. This is done by adding some diagonals as “cuts” and an unfold process. $V_s(q)$ in $\mathcal{P}_s$ is then computed using an algorithm in [12, 13], from which the final $V(q)$ in $\mathcal{P}$ is computed. This is done by a SEE-THROUGH procedure to be described below.

Figure 3 depicts an example of the algorithm. The original non-simple $\mathcal{P}$ and its $\mathcal{P}_s$ are shown in parts A and B respectively. $V(q)$ in $\mathcal{P}_s$, denoted by $V_s(q)$ is computed as shown in C. There are segments in $\mathcal{P}$ that are visible from q through the “cut” segments of $V_s(q)$ which are shown in D. These segments are computed (recursively) by the SEE-THROUGH algorithm to replace the cut-segments of $V_s(q)$ which leads to the final $V(q)$, as shown in E.

3.1 Creating a simple polygon from $\mathcal{P}$

We produce a simple polygon $\mathcal{P}_s$ from $\mathcal{P}$ by eliminating its holes. One hole, say H, in $\mathcal{P}$ can be eliminated by adding a pair of cut-diagonals connecting a vertex of H to a vertex of $\mathcal{P}$ in its outer boundary. Cutting $\mathcal{P}$ along this diagonal produces another polygon in which H is no longer a hole. We continue this process on this new polygon to eliminate all holes. A cut-diagonal should lie completely inside $\mathcal{P}$ and should not intersect any other hole. This can be enforced if we eliminate leftmost hole first, which is the hole whose leftmost corner has smallest x-coordinate.

For $\mathcal{P}$ with total of n vertices and h holes, $\mathcal{P}_s$ will have $n + 2h$ vertices. We know that the upper bound of h is $\lfloor \frac{n-3}{3} \rfloor$. Hence, the number of the vertices of $\mathcal{P}_s$ is also $O(n)$.

The conversion algorithm described above can be done by first triangulating the polygon and then selection the proper cut-diagonals, which can be done in $O(n \log^* n)$ [19].

3.2 Computing visibility through cut-diagonals

As mentioned before, an important step of our approach is the SEE-THROUGH algorithm to update the $V_s(q)$ on $\mathcal{P}_s$. This step finds new segments of edges of $\mathcal{P}$ that are visible from q through the cut-diagonals. We use the ideas presented in [13].

Theorem 2 . [13] *Given a simple polygon P with size n and a cut-diagonal which cuts P into two parts, L and R, by using $O(n^2 \log n)$ time, we could construct a data structure of size $O(n^2)$ so that, for any query point $q \in R$, the partial visibility $V_L(q)$ through the diagonal can be reported in $O(\log n + |V_L(q)|)$ time.*

This theorem does not hold for non-simple polygons. We however, use its idea for polygons with holes. Assume that $\mathcal{P}$ has only one hole H_i which has been eliminated by one cut-diagonal u_1u_2 as shown in Figure 8. For any query point q, we intend to find the set of segments on edges of $\mathcal{P}$ that are visible from q through u_1u_2. Continuing u_1u_2 through H_i will lead to another segment v_1v_2 such that v_1 is on H_i and v_2 is the first encounter of this segment with $\mathcal{P}$. Now suppose that the cut-diagonal u_1u_2 is replaced by v_1v_2. Obviously, cutting $\mathcal{P}$ along v_1v_2 will produce another simple polygon, called P'_{H_i} for which u_1u_2 is a diagonal. We can now use Theorem 2 to preprocess P'_{H_i} and build the appropriate data structures so that for any query point q, we can find the segments of P'_{H_i} that are visible from q through its diagonal u_1u_2. These segments are denoted as $V_{H_i}(q)$. Since, no part of v_1v_2 is visible from q through u_1u_2, $V_{H_i}(q)$ is the same as set of segments we are looking for. Let us denote the above algorithm by SEE-THROUGH(H_i).

This algorithm can be extended to more holes. This is done by performing SEE-THROUGH(H_i) once for each H_i assuming that $\mathcal{P}$ has effectively been cut along the cut-diagonals of other holes, which leads to a polygon with only one hole H_i. Therefore, we have h data structures resulted from these preprocessing steps. Given the query point, we can find the extra segments of P visible from q through all the cut-diagonals, to be described in next subsection.

3.3 The algorithm

The preprocessing phase of the algorithm is done as follows:

a Add all cut-diagonals to produce the simple polygon $\mathcal{P}_s$, as described in subsection 3.1.

b Preprocess $\mathcal{P}_s$ and create the data structure so that $V(q)$ of any arbitrary query q in $\mathcal{P}_s$ can efficiently be reported. Theorem 3 is used for this step.

c For each hole, H_i, perform SEE-THROUGH(H_i) to preprocess the polygon, so that for any query point q, $V_{H_i}(q)$ can be computed efficiently.

Theorem 3 . [12] *A simple polygon P can be preprocessed in $O(n^3 \log n)$ time and $O(n^3)$ space such that given an arbitrary query point q inside the polygon, $O(\log n + |V(q)|)$ time is sufficient to recover $V(q)$.*

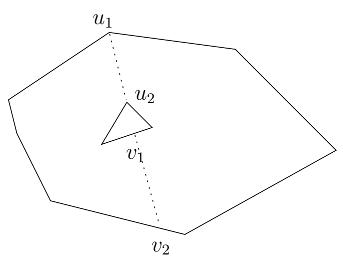

Figure 4: Replacing u_1u_2 with v_1v_2.

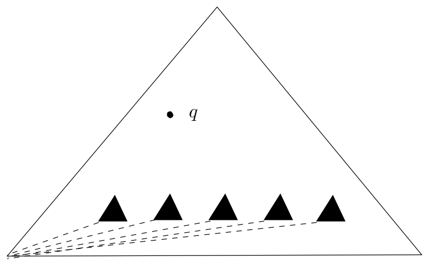

Figure 5: A polygon with tight bound of h'.

The $V(q)$ of any q is computed as follows. The data structure build at step (b) of the preprocessing phase is used to find $V_s(q)$ the set of segments viewed by q in $\mathcal{P}_s$. Suppose that a segment of the cut-diagonal uv of a hole H_i is a part of $V_s(q)$. The preprocessing of step (c) is then used to find $V_{H_i}(q)$, the extra segments viewed through uv. The segment uv in $V_s(q)$ is then replaced by $V_{H_i}(q)$. This is continued for any such segments in $V_s(q)$. This process will finish without any loop, due to the nature of visibility. What remains at the end is $V(q)$, and this is easy to prove.

Lemma 3 . *The preprocessing time and space complexities of the algorithm are $O(n^3 \log n)$ and $O(n^3)$, respectively.*

Proof. Time complexity of the preprocessing of step (a), as described in section 3.1, is $O(n \log^* n)$. Also, the resulted polygon $\mathcal{P}_s$, like $\mathcal{P}$, has $O(n)$ vertices. The time and space complexities of step (b) are as in Theorem 3.

As described in subsection 3.2, the preprocessing time for any cut-diagonal is of size $O(n^2 \log n)$ and the size of its data structure is $O(n^2)$. There are at most $O(n)$ such diagonals in $\mathcal{P}$. Thus, the total preprocessing time to construct cut-diagonal data structures is $O(n^3 \log n)$ and they require $O(n^3)$ space. □

Lemma 4 . *The query time to report $V(q)$ is $O(\log n + h' \log n + h' + |V(q)|)$ where h' is the number of cut-diagonals appeared in $V_s(q)$ during the algorithm.*

Proof. A point location of time $O(\log n)$ is done to find the location of q in $\mathcal{P}_s$. For any one of the h' cut-diagonals, appeared in $V_s(q)$, a point location of size $O(\log n)$ is required to run the SEE-THROUGH algorithm. The number of the edges that appear in $V_s(q)$ is the size of the final visibility polygon $V(q)$ plus the number of the cut-diagonals appeared in $V_s(q)$. □

Lemma 5 . *The upper bound of h' is $O(h^2)$ and this bound is tight.*

Proof. The cut-diagonals do not intersect each other except at their end-points. Therefore, if a query point q sees a cut-diagonal l through another cut-diagonal l' then it is impossible for q to see l' through l. Also, only a single segment of another cut-diagonal can *directly* be seen from a query point through another cut-diagonal. By directly we mean that there is no other intermediary cut-diagonals between them. Hence, the upper bound of h' is $O(h^2)$. Figure 5 shows a sample with tight bound of h'. □

The value of h' for a query point depends on the position of the point and the cut-diagonals, and the upper bound of h' happened rarely. However, h is $O(n)$ and therefore, the upper bound of h' is $O(n^2)$, which is too weak, theoretically. In the next subsection, we will improve the algorithm and overcome this weakness.

3.4 Improving the algorithm

As shown in Figure 5, in some cases, the processing of a cut-diagonal does not directly change the final visibility polygon; the algorithm just moves one step further to another cut-diagonal. We will show that the upper bound of h' occurs only when there are many of these cases. If we

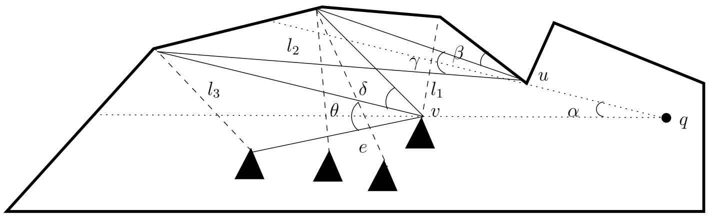

Figure 6: The cut-diagonal e is q-ineffective.

preprocess to skip these cases, the upper bound of h' is shown to be reduced effectively. To do this, we prepare another data structure by which these *ineffective* intermediate cut-diagonals can be skipped. A cut-diagonal e is called *q-effective* if some portion of $\mathcal{P}$ other than just another single cut-diagonal and its endpoints is visible to q through e. Otherwise, it is *q-ineffective*. In Figure 6, e is q-ineffective.

As shown in Figure 6, a query point, like q, views a portion of a cut-diagonal l_1 through an angle α, which is constrained by two reflex vertices. These vertices may be the endpoints of l_1 or reflex vertices of $\mathcal{P}$ that lie between q and l_1. In Figure 6, u and v are the reflex vertices associated to q and l_1.

In our algorithm, l_1 is a segment in $V_s(q)$ which must be replaced by applying SEE-THROUGH. This will replace l_1 by another cut-diagonal e. Applying SEE-THROUGH on e will further replace e by another cut-diagonal l_2. Finally, SEE-THROUGH on l_2 replaces l_2 by l_3 and an edge of $\mathcal{P}$. Therefore, we can instead start from l_2 as if l_1 and e never existed. This is true for all q in the polygon whose visibility range is bounded by u and v and the lines qu and qv extend within γ and θ angles, respectively.

To ignore ineffective cut-diagonals, for any pair of reflex vertices, a data structure is maintained that, for any query point q, efficiently determines the first q-effective cut-diagonal. For each reflex vertex v, we first compute the different angular ranges around it through which an observer sees different segments of $\mathcal{P}$. Parameters β and θ are examples of such ranges for reflex vertex v in Figure 6. These ranges are produced by connecting v to the vertices of $\mathcal{P}$ that are visible from v. These ranges produce a radial decomposition of $\mathcal{P}$ around v that we refer to as RD_v. To avoid degenerate cases, we assume that no three vertices of P are collinear.

For each pair of reflex vertices u and v, another data structure, denoted as $VR_{u,v}$ (for *Visibility Ranges*), is then built over their radial decompositions. In this data structure $VR_{u,v}(\alpha,\beta)$ is the first effective cut-diagonal to all points that see within the ranges α of u and β of v. For each (α,β), that α and β are two ranges of RD_u and RD_v respectively, $VR_{u,v}(\alpha,\beta)$ is computed and maintained.

These data structures are prepared in the preprocessing phase. For any query point q that sees a cut-diagonal through the reflex vertices u and v, we compute the ranges α and β which respectively are the ranges in which the extensions qu and qv lie. Having (α,β), its associated q-effective cut-diagonal, $VR_{u,v}(\alpha,\beta)$, is found from $VR_{u,v}$ and reported as the first q-effective cut-diagonal, which must be processed by the SEE-THROUGH procedure.

Clearly, $\mathcal{P}$ can have $O(n)$ reflex vertices and the size of RD_v for any reflex vertex v can be $O(n)$. Hence, there can be $O(n^2)$ pairs of reflex vertices u and v for which $VR_{u,v}$ must be maintained. Naively, according to the size of RD, any one of the VR data structures can be of size $O(n^2)$, and therefore, the total size of VR's is $O(n^4)$.

We know that the predefined ranges of α and β for any given query point q are divergent. Therefore, the entries of $VR_{u,v}(\alpha,\beta)$ for convergent values of α and β are not important and thus are not maintained. In addition, many of the entries of $VR_{u,v}$ are empty, meaning that l itself is the effective cut-diagonal. Moreover, the values of some entries can be derived from the values of other entries and thus, it is not required to maintain these entries in $VR_{u,v}$.

Lemma 6 . *Only $O(n)$ entries of $VR_{u,v}$ are nonempty and will be maintained. Therefore, the size of $VR_{u,v}$ for any pair of reflex vertices is $O(n)$.*

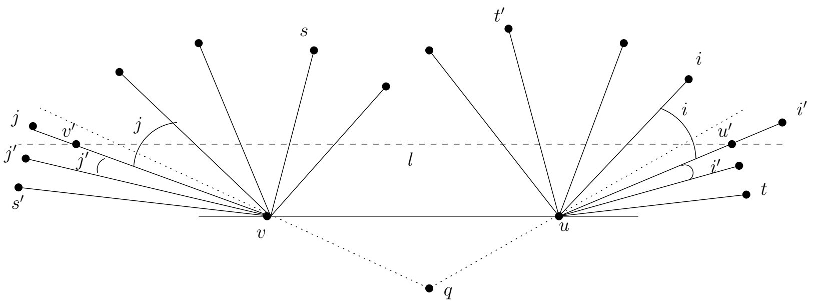

Figure 7: Only $O(n)$ entries of $VR_{u,v}$ are not empty.

Proof. Consider the reflex vertices u and v for which $VR_{u,v}$ is to be constructed. If the segment uv intersects $\mathcal{P}$, then it is impossible for u and v to be the bounding vertices for the vision of a query point. Therefore, it is not required to maintain $VR_{u,v}$ in such situations. So, we assume that this segment in contained completely inside $\mathcal{P}$. The RD ranges of u and v are considered in counter clockwise order and are nominated according to Figure 7. According to this nomination, the range i of vertex u is bounded from above by segment ui in which i is a vertex of $\mathcal{P}$ that is visible to u. However, RD_u has such ranges only for those vertices of $\mathcal{P}$ which are visible to u.

Assume that the entry $VR_{u,v}(i,j)$ is not empty and its value is l. This means that for any query point q, that qu and qv extend within the i and j ranges of RD_u and RD_v, respectively, all of the cut-diagonals that exist between l and uv are q-ineffective.

In addition, assume that i and j are the locally farthest apart so that there is no range i' of RD_u before i with nonempty entry of $VR_{u,v}(i',j)$ and no range j' of RD_v after j with nonempty entry of $VR_{u,v}(i,j')$. These assumptions lead to the following conclusions:

1. The region $uu'v'v$ does not contain any vertex of $\mathcal{P}$ and has no intersection with edges of $\mathcal{P}$. This is a trivial result of existence of l as the value of $VR_{u,v}(i,j)$.
2. For any range s of v lying before j and any range t' of u that exists after i such that s and t' are not convergent, $VR_{u,v}(t',s)$ is equal to l or another cut-diagonal l' that lies farther than l from uv. If the value of this entry of $VR_{u,v}$ is equal to l, then there is no need to be maintained and can be induced from the value of $VR_{u,v}(i,j)$. We do not maintain such unnecessary entries in $VR_{u,v}$ and leave them empty. When we search the value of $VR_{u,v}(\alpha,\beta)$ for nonconvergent α and β, the value of $VR_{u,v}(i,j)$ is returned in which i is the smallest range of u lying before α and j is the greatest range of v lying after β.
3. For any range s before j and ranges t and t' before and after range i respectively, only one of the $VR_{u,v}(t,s)$ and $VR_{u,v}(t',s)$ can be nonempty. The reason is that the entry $VR_{u,v}(t',s)$ can be nonempty only if s exists on the right of the range i (see Figure 7). If the entry $VR_{u,v}(t,s)$ is also nonempty with value l', that is distinct from l, because of the maximality of l, then l and l' must intersect or the right end-point of l, that is a vertex of $\mathcal{P}$, must exist between l' and uv. However, none of these conditions is possible.
4. For any range t' after i and ranges s and s' before and after range j respectively, only one of the $VR_{u,v}(t',s)$ and $VR_{u,v}(t',s')$ can be nonempty. The proof is the same as above.
5. For any pair of ranges i and i' of u there is at most one range j of v such that both of $VR_{u,v}(i,j)$ and $VR_{u,v}(i',j)$ are nonempty. This is a trivial result of the above two items.

The last conclusion implies that there are at most $2n$ nonempty entries in $VR_{u,v}$ and thus the lemma is correct. □

Consider a view point that lies on vertex u and its vision range is from angle α to β. We define $VR_u(\alpha,\beta)$ to be the first effective cut-diagonal to such a view point. VR_u can be computed using our algorithm. For the vertex u, $V(u)$ is computed and during this computation when a cut-diagonal appears in $V(u)$ its VR_u is updated accordingly. Moreover, RD_u can also be

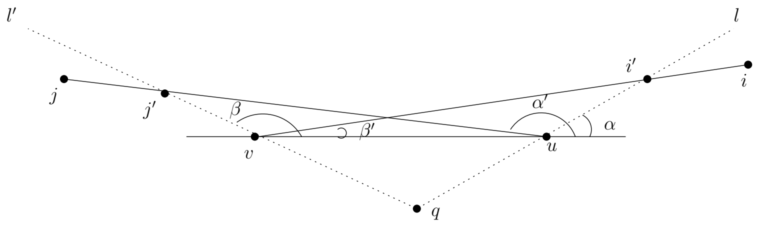

Figure 8: $VR_{u,v}(\alpha,\beta)$ is the same as $VR_u(\alpha,\alpha')$ and $VR_v(\beta',\beta)$.

constructed as a byproduct of this process. Size of a VR_u structure is also $O(n)$ by the same argument presented for $VR_{u,v}$.

Assume that a query point q sees within the reflex vertices u and v with visibility ranges of α and β according to Figure 8. Also, assume that β' is the greatest visibility range of vertex v with an endpoint i that lies before line l, such that there is no vertex in the vui' region. Moreover, assume that α' is the smallest visibility range of u with an endpoint j that lies after the line l', such that there is no vertex in the vuj' region (See Figure 8). We claim that $VR_{u,v}(\alpha,\beta)$ is the same as $VR_u(\alpha,\alpha')$ and $VR_v(\beta',\beta)$.

Definition of i and j forces equality of $VR_u(\alpha,\alpha')$ and $VR_v(\beta',\beta)$. If they are not equal, then there must exist a vertex that is visible only from one of u and v. That vertex must be contained in $ii'l$ or $jj'l'$ regions which is impossible. A similar argument implies equality of $VR_{u,v}(\alpha,\beta)$ and $VR_u(\alpha,\alpha')$ or $VR_v(\beta',\beta)$.

Therefore, we can think of $VR_{u,v}$ as the composition of VR_u and the α' ranges that are computed with respect to v. These α' ranges are maintained in a data structure called $VR'_{u,v}$. For any query point q that sees within the reflex vertices u and v with visibility ranges α and β respectively, instead of using $VR_{u,v}$, we first find α' from $VR'_{u,v}$, associated to range β and then report $VR_u(\alpha,\alpha')$ as the first q-effective cut-diagonal. Also, we can use $VR'_{v,u}$ and VR_v instead of $VR'_{u,v}$ and VR_u.

For any pair of reflex vertices u and v, we use a radial sweep line around v to find α' ranges and another one around u to find β' ranges. These ranges are maintained as sorted lists in $VR'_{u,v}$ and $VR'_{v,u}$.

Lemma 7 . *Maintaining a data structure of size $O(n^3)$ which can be constructed in $O(n^3 \lg n)$ time makes it possible to skip ineffective cut-diagonals and find the first effective one in $O(\lg n)$ time.*

Proof. Size of VR_u for any reflex vertex u is $O(n)$ and can pessimistically be constructed by using our visibility algorithm in time $O(n^2 \lg n)$. Also, RD_u with size $O(n)$ can be built as a byproduct of this process. Computing $VR'_{u,v}$ for any pair of reflex vertices u and v can be done in $O(n \lg n)$ time by the radial sweep line algorithm on RD_v ranges. Size of $VR'_{u,v}$ is also $O(n)$. There are $O(n^2)$ pairs of reflex vertices. Hence, the total time and space required to build and maintain $VR'_{u,v}$ structures are $O(n^3 \lg n)$ and $O(n^3)$ respectively.

To find the first effective cut-diagonal of a query point, the $VR'_{u,v}$ and VR_u data structures must be searched, both requires $O(\lg n)$ time. Therefore, the query time of reporting the first effective cut-diagonal is $O(\lg n)$. □

In above discussion we ignored the computation of $VR_{u,v}$. However, these structures can be constructed by a divide and conquer algorithm on VR_u and $VR'_{u,v}$ structures or VR_v and $VR'_{v,u}$. This can be done as a preprocess that results in a constant factor reduction of the memory usage and query time.

Lemma 8 . *Number of the effective cut-diagonals that will be processed by the algorithm for a query point q is $O(|V(q)|)$. Thus, the upper bound of the parameter h' is $O(|V(q)|)$.*

Proof. Any effective cut-diagonal adds at least one edge or vertex to the visibility polygon of the query point and no one of these changes are repeated twice. Therefore, an upper bound of the number of these cut-diagonals is the size of the visibility polygon. □

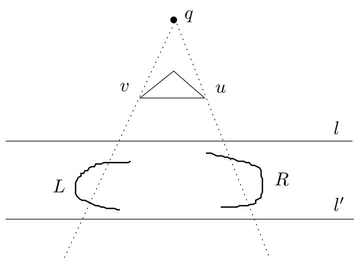

Figure 9: A hole can produce only two q-effective cut-diagonals.

The above lemma decreases the query time of the algorithm to $O(\log n + |V(q)| \log n + |V(q)|)$.

Lemma 9 . *The number of the effective cut-diagonals that will be processed by the algorithm for a query point q is $O(h)$. Therefore, the upper bound of the parameter h' is $O(h)$.*

Proof. Clearly, a viewpoint could see at most h cut-diagonals directly. If a viewpoint see two parts of a cut-diagonal, at least one of them must be seen via another cut-diagonal. Thus, a hole must be exist between that diagonal and the viewpoint. According to Figure 9, the point q sees two parts of the cut-diagonal l and l'. This is due to the hole above them. If the left part of the edge l is q-effective, then there must be some vertices of the polygon forming a chain, like L in Figure 9.

Also, if the right part of the edge l is q-effective, then a chain, similar to R in Figure 9 must exist. If both of these chains exist and the viewpoint see the left and right parts of the cut-diagonal l', then L and R must compose another hole. Hence, we can say that the two parts of the cut-diagonal l' are effective because of this hole. Therefore, the number of the effective segments of the cut-diagonals is at most $2h$. □

Theorem 4 . *A non-simple polygon $\mathcal{P}$ with a total n vertices and h holes inside $\mathcal{P}$, can be preprocessed in time $O(n^3 \log n)$, and maintaining a data structure of size $O(n^3)$, so that the visibility polygon of an arbitrary query point q within $\mathcal{P}$ can be reported in time $O((1+h') \log n + |V(q)|)$ in which $|V(q)|$ is the size of the output and h' is an output and preprocessing sensitive parameter of size at most* $\min(h, |V(q)|)$.

4 Conclusion

In this paper, we have considered the problem of computing the visibility polygon $V(q)$ of a point q inside a polygon with holes. This problem has been solved efficiently before, but in the non-query version. Here, we proposed an efficient algorithm for the query version of the problem where we preprocess the polygon to build a data structure by which the $V(q)$ of any query point q could be reported rapidly.

We first applied and analyzed the notion of visibility decomposition to this type of polygons. We then presented an algorithm to report $V(q)$ of any q in time $O((1 + h') \log n + |V(q)|)$ by spending $O(n^3 \log n)$ time to preprocess the polygon and maintaining a data structure of size $O(n^3)$. The factor h' is an output and preprocess sensitive parameter of size at most $\min(h, |V(q)|)$.

It is interesting to know if this method can be used to solve other similar problems such as finding the visibility polygon of a moving point and a line segment or the visibility graph of a point set especially in dynamic environments. Another question is whether we can omit the factor h' or reduce it to $O(\lg h')$.

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